Gale duality for complete intersections

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(joint work with Frédéric Bihan)

This talk is based upon the preprint [4]. A complete intersection in $(\mathbb{C}^\times)^{n+m}$ defined by Laurent polynomials,

$$f_1(x_1,\ldots,x_{m+n}) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_{m+n}) \;=\; 0\,, \tag{1}$$

where each polynomial f_i contains the same monomials $\{1, x^{\alpha_1}, \ldots, x^{\alpha_{l+m+n}}\}$ may also be viewed as the intersection of a codimension n affine linear space L in $\mathbb{C}^{l+m+n}$ with the image of $(\mathbb{C}^\times)^{m+n}$ under the map

$$\varphi \,:\, (\mathbb{C}^\times)^{m+n} \;\ni\; x \;\longmapsto\; (x^{\alpha_1},\ldots,x^{\alpha_{l+m+n}}) \in (\mathbb{C}^\times)^{l+m+n} \subset \mathbb{C}^{l+m+n}\,.$$

When the exponent vectors $\{\alpha_1,\ldots,\alpha_{l+m+n}\}$ span the integer lattice $\mathbb{Z}^{m+n}$, the map φ is injective and the complete intersection (1) in $(\mathbb{C}^\times)^{m+n}$ is scheme-theoretically isomorphic to the intersection $\varphi((\mathbb{C}^\times)^{m+n}) \cap L$.

Suppose that $\psi\colon \mathbb{C}^{l+m} \to L$ parameterizes L. Then $\psi^{-1}(\varphi((\mathbb{C}^\times)^{m+n}) \cap L)$ is also isomorphic to the original complete intersection (1). In the coordinates for $\mathbb{C}^{l+m+n}$, ψ is given by degree 1 polynomials $p_1(y),\ldots,p_{l+m+n}(y)$, and the inverse image of $(\mathbb{C}^\times)^{l+m+n}$ is the complement M_H of the arrangement H of hyperplanes in $\mathbb{C}^{l+m}$ defined by $\prod_i p_i(y) = 0$. If $z_1,\ldots,z_{l+m+n}$ are coordinates for $\mathbb{C}^{l+m+n}$, then $\varphi((\mathbb{C}^\times)^{m+n})$ is defined in $(\mathbb{C}^\times)^{l+m+n}$ by all monomial equations $z^\beta = 1$, where $\beta = (b_1,\ldots,b_{l+m+n}) \in \mathbb{Z}^{l+m+n}$ is a vector such that

$$b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{l+m+n}\alpha_{l+m+n} \;=\; 0\,.$$

The monomial z^β pulls back to a *master function* on M_H,

$$p(y)^\beta \;:=\; (p_1(y))^{b_1}\cdot(p_2(y))^{b_2}\cdots(p_{l+m+n}(y))^{b_{l+m+n}}\,.$$

Letting $\beta_1,\ldots,\beta_l$ form a basis for the free abelian group of all such linear relations, we see that the pullback $\psi^{-1}(\varphi((\mathbb{C}^\times)^{m+n}) \cap L)$ is a complete intersection in M_H defined by the system of master functions,

$$p(y)^{\beta_1} \;=\; p(y)^{\beta_2} \;=\; \cdots \;=\; p(y)^{\beta_l} \;=\; 1\,. \tag{2}$$

We say that the system of polynomials (1) in $(\mathbb{C}^\times)^{m+n}$ is *Gale dual* to the system of master functions (2) in M_H.

The isomorphism between schemes defined by Gale dual systems was a key idea behind the new fewnomial bounds in [1, 2, 3]. The number of positive solutions of a system of n polynomials in n variables with $l+n+1$ monomials is at most

$$\tfrac{e^2+3}{4}2^{\binom{l}{2}}n^l\,.$$

This dramatically improves Khovanskii's bound [5], which is $2^{\binom{l+n}{2}}(n{+}1)^{l+n}$.

We close with an example. Let $n = l = 2$ and $m = 0$ and consider the system

$$\begin{aligned} x^3y^2 &= x^4y^{-1} - x^4y - \tfrac{1}{2}\,,\\ xy^2 &= x^4y^{-1} + x^4y - 1\,. \end{aligned} \tag{3}$$

in $(\mathbb{C}^\times)^2$. This is isomorphic to $\varphi((\mathbb{C}^\times)^2)\cap L$, where L is defined by

$$z_1-(z_3-z_4-\tfrac{1}{2}) \;=\; z_2-(z_3+z_4-1) \;=\; 0\,,\quad\text{and}$$
$$\varphi\colon\ (x,y)\ \longmapsto\ (x^3y^2, xy^2, x^4y^{-1}, x^4y) \;=\; (z_1,z_2,z_3,z_4)\,.$$

Let s,t be new variables and set

$$p_1 \;:=\; s-t-\tfrac{1}{2} \qquad\qquad p_3 \;:=\; s$$
$$p_2 \;:=\; s+t-1 \qquad\qquad p_4 \;:=\; t$$

Then $\psi_p\colon (s,t)\mapsto(p_1,p_2,p_3,p_4)$ parametrizes L.

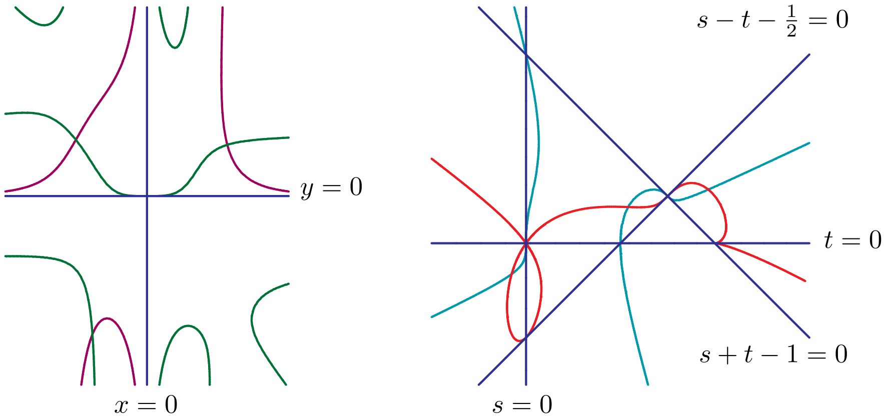
The primitive weights $(-1,3,2,-2)$ and $(3,-1,1,-3)$ annihilate the exponents:

$$(x^3y^2)^{-1}(xy^2)^3(x^4y^{-1})^2(x^4y)^{-2} \;=\; (x^3y^2)^3(xy^2)^{-1}(x^4y^{-1})(x^4y)^{-3} \;=\; 1\,.$$

The polynomial system (3) in $(\mathbb{C}^\times)^2$ is equivalent to the system of master functions

$$\frac{s^2(s+t-1)^3}{t^2(s-t-\frac{1}{2})} \;=\; \frac{s(s-t-\frac{1}{2})^3}{t^3(s+t-1)} \;=\; 1\,. \tag{4}$$

in the complement of the hyperplane arrangement $st(s+t-1)(s-t-\frac{1}{2})=0$.



The polynomial system (3) and the system of master functions (4).

References

[1] D.J. Bates, F. Bihan, and F. Sottile, *Bounds on real solutions to polynomial equations*, IMRN, (2007), 2007:rnm114-7.

[2] F. Bihan, J.M. Rojas, and F. Sottile, *Sharpness of fewnomial bounds and the number of components of a fewnomial hypersurface*, Algorithms in Algebraic Geometry (A. Dickenstein, F.-O. Schreyer, and A. Sommese, eds.), IMA Volumes in Mathematics and its Applications, vol. 146, Springer New York, 2007, pp. 15–20.

[3] F. Bihan and F. Sottile, *New fewnomial upper bounds from Gale dual polynomial systems*, Moscow Mathematical Journal **7** (2007), no. 3, 387–407.

[4] ———, *Gale duality for complete intersections*, 2007, Annales de l'Institut Fourier, to appear.

[5] A.G. Khovanskii, *Fewnomials*, Trans. of Math. Monographs, 88, AMS, 1991.