# ORBITOPES

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Abstract. An orbitope is the convex hull of an orbit of a compact group acting linearly on a vector space. These highly symmetric convex bodies lie at the crossroads of several fields including convex geometry, algebraic geometry, and optimization. We present a self-contained theory of orbitopes with particular emphasis on instances arising from the groups $SO(n)$ and $O(n)$. These include Schur-Horn orbitopes, tautological orbitopes, Carathéodory orbitopes, Veronese orbitopes, and Grassmann orbitopes. We study their face lattices, their algebraic boundaries, and representations as spectrahedra or projected spectrahedra.

## 1. Introduction

An *orbitope* is the convex hull of an orbit of a compact algebraic group $G$ acting linearly on a real vector space. The orbit has the structure of a real algebraic variety, and the orbitope is a convex, semi-algebraic set. Thus, the study of algebraic orbitopes lies at the heart of *convex algebraic geometry* – the fusion of convex geometry and (real) algebraic geometry.

Orbitopes have appeared in many contexts in mathematics and its applications. Orbitopes of finite groups are highly symmetric convex polytopes which include the platonic solids, permutahedra, Birkhoff polytopes, and other favorites from Ziegler's text book [38], as well as the Coxeter orbihedra studied by McCarthy, Ogilvie, Zobin, and Zobin [25]. Farran and Robertson's regular convex bodies [11] are orbitopal generalzations of regular polytopes, which were classified by Madden and Robertson [24]. Orbitopes for compact Lie groups, such as $SO(n)$, have appeared in investigations ranging from protein structure prediction [23] and quantum information [2] to calibrated geometries [16]. Barvinok and Blekherman studied the volumes of convex bodies dual to certain $SO(n)$-orbitopes, and they concluded that there are many more non-negative polynomials than sums of squares [4].

This paper initiates the study of orbitopes as geometric objects in their own right. The questions we ask about orbitopes originate from three different perspectives: *convexity*, *algebraic geometry*, and *optimization.* In convexity, one would seek to characterize all faces of an orbitope. In algebraic geometry, one would examine the Zariski closure of its boundary and identify the components and singularities of that hypersurface. In optimization, one would ask whether the orbitope is a spectrahedron or the projection of a spectrahedron.

Spectrahedra are to semidefinite programming what polyhedra are to linear programming. More precisely, a *spectrahedron* is the intersection of the cone of positive semidefinite matrices

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with an affine space. It can be represented as the set of points $x \in \mathbb{R}^n$ such that

$$A_0 + x_1 A_1 + \cdots + x_n A_n \;\succeq\; 0\,, \tag{1.1}$$

where $A_0, A_1, \ldots, A_n$ are symmetric matrices and $\succeq 0$ denotes positive semidefiniteness. From a spectrahedral description many geometric properties, both convex and algebraic, are within reach. Furthermore, if an orbitope admits a representation (1.1) then it is easy to maximize or minimize a linear function over that orbitope. Here is a simple illustration.

**Example 1.1.** Consider the action of the group $G = SO(2)$ on the space $\mathrm{Sym}_4(\mathbb{R}^2) \simeq \mathbb{R}^5$ of binary quartics and take the convex hull of the orbit of $v = x^4$. The four-dimensional convex body $\mathrm{conv}(G \cdot v)$ is a *Carathéodory orbitope*. This orbitope is a spectrahedron: it coincides with the set of all binary quartics $\lambda_0 x^4 + 4\lambda_1 x^3 y + 6\lambda_2 x^2 y^2 + 4\lambda_3 x y^3 + \lambda_4 y^4$ such that

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix} \succeq 0 \qquad \text{and} \qquad \lambda_0 + 2\lambda_2 + \lambda_4 = 1. \tag{1.2}$$

This representation (1.2) will be derived in Section 5, where we will also see that it is equivalent to classical results from the theory of positive polynomials [32]. The Hankel matrix shows that the boundary of $\mathrm{conv}(G \cdot v)$ is an irreducible cubic hypersurface in $\mathbb{R}^4$, defined by the vanishing of the Hankel determinant. It also reveals that this four-dimensional Carathéodory orbitope is 2-neighborly: the extreme points are the rank one matrices, and any two of them are connected by an edge. The typical intersection of $\mathrm{conv}(G \cdot v)$ with a three-dimensional affine plane looks like an inflated tetrahedron. This three-dimensional convex body is bounded by *Cayley's cubic surface*, shown in Figure 1. Alternative pictures of this convex body can be found in [27, Fig. 3] and [35, Fig. 4]. The four vertices of the tetrahedron lie on the curve $G \cdot v$, and its six edges are inclusion-maximal faces of $\mathrm{conv}(G \cdot v)$. □

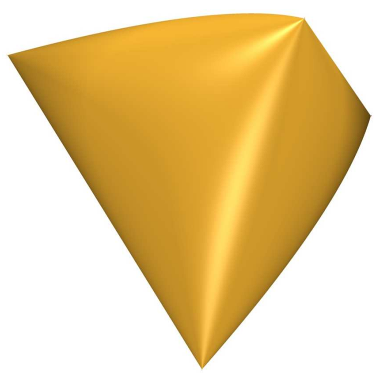


FIGURE 1. Cross-section of a four-dimensional Carathéodory orbitope.

This article is organized as follows. We begin by deriving the basic definitions and a few general results about orbitopes, and we formulate **ten key questions** which will guide our subsequent investigations. These are organized along the themes of convex geometry

(Subsection 2.1), algebraic geometry (Subsection 2.2) and optimization (Subsection 2.3). These questions are difficult. They are meant to motivate the reader and to guide our investigations. We are not yet able to offer a complete solution to any of these ten problems.

Section 3 is concerned with the action of $O(n)$ and $SO(n)$ by conjugation on $n\times n$-matrices. These decompose into the actions by conjugation on symmetric and skew-symmetric matrices. The resulting *Schur-Horn orbitopes* are shown be spectrahedra, their algebraic boundary is computed, and their face lattices are derived from certain polytopes known as permutahedra. The spectrahedral representations of the Schur-Horn orbitopes, stated in Theorems 3.4 and 3.15, rely on certain Schur functors also known as the *additive compound matrices*.

Given a compact real algebraic group of $n\times n$-matrices, we can form its convex hull in $\mathbb{R}^{n\times n}$. The resulting convex bodies are the *tautological orbitopes*. In Subsection 4.2 we study the tautological orbitope and its dual coorbitope for the group $O(n)$. Both are spectrahedra, characterized by constraints on singular values, and they are unit balls for the operator and nuclear matrix norm considered in [31]. Subsections 4.1 and 4.4 are devoted to the tautological orbitope for $SO(n)$. A characterization of its faces is given in Theorem 4.11.

The orbitopes of the group $SO(2)$ are the convex hulls of trigonometric curves, a classical topic initiated by Carathéodory in [8], further developed in [5, 34], and our primary focus in Section 5. These $SO(2)$-orbitopes can be represented as projections of spectrahedra in two distinct ways: in terms of Hermitian Toeplitz matrices or in terms of Hankel matrices.

A natural generalization of the rational normal curves in Section 5 are the Veronese varieties. Their convex hulls, the *Veronese orbitopes*, are dual to the cones of polynomials that are non-negative on $\mathbb{R}^n$, as seen in [4, 7, 32]. In Section 6 we undertake a detailed study of the 15-dimensional Veronese orbitope and its coorbitope which arise from ternary quartics.

In Section 7 we mesh our investigations with a line of research in differential geometry. The *Grassmann orbitope* is the convex hull of the oriented Grassmann variety in its Plücker embedding in the unit sphere in $\wedge_d\mathbb{R}^n$. This vector space is the $d$-th exterior power of $\mathbb{R}^n$ and it is isomorphic to $\mathbb{R}^{\binom{n}{d}}$. The facial structure of Grassmann orbitopes has been studied in the theory of calibrated manifolds [16, 17, 26]. We take a fresh look at these orbitopes from the point of view of convex algebraic geometry. Theorem 7.3 furnishes a spectrahedral representation and the algebraic boundary in the special case $d = 2$, while Theorem 7.6 shows that the Grassmann orbitopes fail to be spectrahedra in general.

## 2. Setup, Tools and Questions

Let $G$ be a real, compact, linear algebraic group, that is, a compact subgroup of $GL(n,\mathbb{R})$ for some $n \in \mathbb{N}$ given as a subvariety. Prototypic examples are the *special orthogonal group* $SO(n) = \{g \in GL(n,\mathbb{R}) : gg^T = \mathrm{Id}_n \text{ and } \det(g) = 1\}$ and the *unitary group* $U(n) = \{g \in GL(n,\mathbb{C}) : g\overline{g}^T = \mathrm{Id}_n\}$. A real representation of $G$ is a group homomorphism $\rho : G \to GL(V)$ for some finite dimensional real vector space $V$. We will write $g \cdot v := \rho(g)(v)$ for $g \in G$ and $v \in V$. The representation is *rational* if $\rho$ is a rational map of algebraic varieties. By choosing an inner product $\langle \cdot\,,\, \cdot \rangle$ on $V$, we may define a $G$-invariant inner product as

$$\langle v, w\rangle_G \;:=\; \int_G \langle g\cdot v, g\cdot w\rangle\, d\mu. \tag{2.1}$$

Here $\mu$ denotes the *Haar measure*, which is the unique $G$-invariant probability measure on $G$. Choosing coordinates on $V$ so that $\langle\cdot,\cdot\rangle_G$ becomes the standard inner product on $V \simeq \mathbb{R}^m$, we can identify the matrix group $G$ with a subgroup of the *orthogonal group* $O(m) = \{g \in GL(m, \mathbb{R}) : gg^T = \mathrm{Id}_m\}$.

The *orbit* of a vector $v \in V$ under the compact group $G$ is the set $G\cdot v = \{g \cdot v : g \in G\}$. This is a bounded subset of $V$. The orbit $G\cdot v$ is a smooth, compact real algebraic variety of dimension

$$\dim G\cdot v \;=\; \dim G - \dim \mathrm{stab}_G(v).$$

Here $\mathrm{stab}_G(v) = \{g \in G : g\cdot v = v\}$ is the *stabilizer* of the vector $v$. In particular, the orbit is isomorphic as a $G$-variety to the compact homogeneous space $G/\mathrm{stab}_G(v)$.

The *orbitope* of $G$ with respect to the vector $v \in V$ is the semialgebraic convex body

$$\mathrm{conv}(G\cdot v) \;=\; \mathrm{conv}\{g\cdot v \,:\, g\in G\} \;\subset\; V.$$

We tacitly assume that the group $G$ and its representation $\rho$ are clear from the context and sometimes we write $\mathcal{O}_v$ or $\mathcal{O}$ for $\mathrm{conv}(G\cdot v)$. The dimension of an orbitope $\mathrm{conv}(G\cdot v)$ is the dimension of the affine hull of the orbit $G\cdot v$.

For small $n$, the connected subgroups of the orthogonal group $O(n)$ are known. This leads to the following census of low-dimensional orbitopes of connected groups.

**Example 2.1** (The orbitopes of dimension at most four arising from connected groups)**.** We identify $G$ with a subgroup of $SO(n)$. We assume $\mathcal{O} = \mathrm{conv}(G\cdot v)$ is an $n$-dimensional orbitope in $\mathbb{R}^n$. This implies that $G$ fixes no non-zero vector. Here is our census:

$n = 1$*:* There are no one-dimensional orbitopes because $SO(1)$ is a point.

It is known that every proper connected subgroup of $SO(2)$ or $SO(3)$ fixes a non-zero vector, so for $n \le 3$, the subgroup $G$ must be equal to $SO(n)$. This establishes the next two cases:

$n = 2$*:* The only orbitopes in $\mathbb{R}^2$ are the discs $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\}$.

$n = 3$*:* The only orbitopes in $\mathbb{R}^3$ are the balls $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le r^2\}$.

The four-dimensional case is where things begin to get interesting:

$n = 4$*:* The group $SO(4)$ has connected subgroups $G$ of dimension 1, 2, 3 and 6.

- If $\dim(G) = 6$ then $G = SO(4)$ and the orbitopes are balls in $\mathbb{R}^4$.
- If $\dim(G) = 3$ then $G \simeq SU(2)$, acting as the unit quaternions on all quaternions, $\mathbb{H} = \mathbb{R}^4$, by either left or right multiplication. Here the orbitopes are also balls in $\mathbb{R}^4$.
- If $\dim(G) = 2$ then $G \simeq SO(2) \times SO(2)$. These tori act on $\mathbb{R}^4$ through an orthogonal direct sum decomposition $\mathbb{R}^2 \oplus \mathbb{R}^2$ and their orbitopes are products of two discs.
- If $\dim(G) = 1$ then $G \simeq SO(2)$ and we obtain four-dimensional orbitopes that are isomorphic, for some positive integers $a < b$, to the Carathéodory orbitopes

$$\mathcal{C}_{a,b} \;:=\; \mathrm{conv}\left\{(\cos at, \sin at, \cos bt, \sin bt) \in \mathbb{R}^4 \mid t \in [0, 2\pi]\right\}.$$

These orbitopes were introduced one century ago by Carathéodory [8]. Their study was picked up in the 1980s by Smilansky [34] and recently by Barvinok and Novik [5]. We note that $\mathcal{C}_{1,2}$ is affinely isomorphic to the Hankel orbitope in Example 1.1. □

To compute the dimensions of orbitopes in general we shall need a pinch of representation theory [13]. A representation $V$ of the group $G$ is *irreducible* if its only subrepresentations are $\{0\}$ and $V$. If $V$ and $W$ are irreducible representations, then the space $\mathrm{Hom}_G(V,W)$ of equivariant linear maps between them is zero unless $V \simeq W$. Schur's Lemma states that $\mathrm{End}_G(V) := \mathrm{Hom}_G(V,V)$ is a division algebra over $\mathbb{R}$, that is, either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. If $W_1, W_2, \ldots$ is a complete list of distinct irreducible representations of $G$, and $V$ is any representation of $G$, then we have a canonical decomposition into isotypical representations:

$$\bigoplus_{i\geq 0} \mathrm{Hom}_G(W_i, V) \otimes_{\mathrm{End}_G(W_i)} W_i \ \xrightarrow{\ \simeq\ }\ V. \tag{2.2}$$

This is an isomorphism of $G$-modules. The map (2.2) on each summand is $(\varphi, w) \mapsto \varphi(w)$. The image of the $i$th summand in $V$ is called the *$W_i$-isotypical component of $V$*, and when it is non-zero, we say that the irreducible representation $W_i$ *appears* in the $G$-module $V$. We say that $V$ is *multiplicity-free* if each irreducible representation $W_i$ appears in $V$ at most once, so that $\mathrm{Hom}_G(W_i, V)$ has rank 1 or 0 over $\mathrm{End}_G(W_i)$.

Suppose that $V$ contains the trivial representation and write $V = \mathbb{R}^l \oplus V'$, where $\mathbb{R}^l$ is the trivial isotypical component of $V$ and $V'$ does not contain the trivial representation. Any vector $v \in V$ can be written as $v = v_0 \oplus v'$, where $v_0 \in \mathbb{R}^l$ and $v' \in V'$. Then

$$G\cdot v \ = \ v_0 \oplus G\cdot v' \qquad \text{and} \qquad \mathrm{conv}(G\cdot v) \ = \ v_0 \oplus \mathrm{conv}(G\cdot v')\,.$$

Thus we lose no geometric information in assuming that $V$ does not contain the trivial representation. This property ensures that the linear span of an orbit coincides with its affine span. Hence the affine span of an orbitope decomposes along its isotypical components:

$$\mathrm{aff}(G\cdot v) \ = \ \bigoplus_{i\geq 0} \mathrm{aff}(G\cdot v_i) \qquad \text{for vectors } \ v = \oplus_i v_i \in V \ \text{ as in (2.2).}$$

To determine the dimension of $\mathrm{aff}(G\cdot v)$ for $v$ in a single isotypical component $V$ we proceed as follows. Let $V = W^l$ with $W$ irreducible. Then $v = (w_1, \ldots, w_l)$ and the affine span of $G\cdot v$ is isomorphic to $W^k$, where $k$ is the rank of the $\mathrm{End}_G(V)$-module spanned by $w_1, \ldots, w_l$. Hence, the dimension of $\mathrm{aff}(G\cdot v)$ over $\mathbb{R}$ equals $k\cdot \dim W$. In particular, $k \leq \dim_{\mathrm{End}_G(V)}(W)$. If $V$ is multiplicity free and $v$ has a nonzero projection into each isotypical component of $V$, then $\dim \mathrm{conv}(G\cdot v) = \dim V$.

We see this in the Carathéodory orbitopes for the group $SO(2)$ of $2\times 2$ rotation matrices. Its nontrivial representations are $W_a \simeq \mathbb{R}^2$, where a rotation matric acts through its $a$th power, for $a > 0$. Let $\mathcal{C}_{a,b}$ be the orbitope of $SO(2)$ with respect to a general vector $v \in W_a \oplus W_b$. If $a \neq b$, then $V$ has two isotypical components and $\mathcal{C}_{a,b}$ has dimension four. If $a = b$, then $V \simeq W_a^2$ consists of a single isotypical component and $\mathcal{C}_{a,a}$ is two-dimensional, as $\mathrm{End}_{SO(2)}(\mathbb{R}^2) = \mathbb{C}$, the span of $(w_1, w_2)$ is complex one-dimensional.

2.1. **Convex geometry.** Orbitopes are convex bodies, and it is natural to begin their study from the perspective of classical convexity. A point $p$ in a convex body $K \subset V$ is an *extreme point* if $\mathrm{conv}(K \backslash \{p\}) \neq K$. Thus, the set $E$ of extreme points of $K$ is the minimal subset satisfying $\mathrm{conv}(E) = K$. An extrinsic description of $K$ is given by its *support function*

$$h(K, \cdot\,) : V^* \to \mathbb{R},\ \ell \mapsto h(K, \ell) := \max\{\ell(x) : x \in K\}.$$

In terms of the support function, the convex body $K$ is the set of points $x \in V$ such that $\ell(x) \leq h(K; \ell)$ for every $\ell \in V^*$. Each linear functional $\ell \in V^*$ defines an *exposed face* of $K$:

$$K^\ell \ = \ \{\, p \in K : \ell(p) = h(K; \ell)\, \}.$$

An exposed face $K^\ell$ is itself a convex body of dimension $\dim \mathrm{aff}(K^\ell)$. An exposed face of dimension 0 is called an *exposed point* of $K$. It follows that every exposed point is extreme, but the inclusion is typically strict. However, for orbitopes, these two notions coincide.

**Proposition 2.2.** *Every point in the orbit* $G \cdot v$ *is exposed in its convex hull. In particular, every extreme point of the orbitope* $\mathrm{conv}(G \cdot v)$ *is an exposed point.*

*Proof.* Since $G$ acts orthogonally on $V$, the orbit $G \cdot v$ lies entirely in the sphere in $V$ that is centered at 0 and contains the point $v$. As every point of the sphere is exposed, the entire orbit consists of exposed points and hence extreme points. □

A closed subset $F \subseteq K$ is a *face* if $F$ contains the two endpoints to any open segment in $K$ it intersects. This includes $\emptyset$ and $K$ itself. An inclusion-maximal proper face of $K$ is called a *facet*. By separation, every face is contained in an exposed face and thus facets are automatically exposed. In general, every exposed face is a face but not conversely.

**Question 1.** *When does an orbitope have only exposed faces?*

The exposed faces of a convex body form a partially ordered set with respect to inclusion, called the *face lattice*. The face lattice is atomic but in general not coatomic as was pointed out to us by Stephan Weis. A sufficient condition is that the polar body (see below) has all faces exposed (cf. [37]). Also, it is generally not graded because "being an exposed face of" is not a transitive relation. For example, the four-dimensional Barvinok-Novik orbitope in Section 5.1 has triangular exposed faces for which the three vertices are exposed but the edges are not. Similar behavior is seen in the convex body on the right of Figure 3, which has two triangular exposed facets whose edges and two of three vertices are not exposed.

**Question 2.** *Describe the face lattices of orbitopes.*

For an orbitope $\mathcal{O} = \mathrm{conv}(G \cdot v)$, the faces come in $G$-orbits and these $G$-orbits come in algebraic families. In particular, the zero-dimensional faces come in a family parametrized by $G$. The description of these families is the point of Question 2. For instance, the orbitope in Example 1.1 is a four-dimensional, two-neighborly convex body. Its exposed points are parametrized by the circle $\mathbb{S}^1$ and the edges come in a two-dimensional family.

The *polar body*

$$\mathcal{O}^\circ \ = \ \{\ell \in V^* : h(\mathcal{O}; \ell) \leq 1\}$$

is called the *coorbitope* of $G$ with respect to $v \in V$. Our assumption that $V$ does not contain the trivial representation ensures that 0 is the centroid of $\mathcal{O}$, and this implies $(\mathcal{O}^\circ)^\circ = \mathcal{O}$. We shall also make use of the cone over the coorbitope $\mathcal{O}^\circ$. This is the *coorbitope cone*

$$\widehat{\mathcal{O}^\circ} \;=\; \big\{(\ell, m) \in V^* \oplus \mathbb{R} \,:\, h(\mathcal{O};\ell) \leq m\big\}. \tag{2.3}$$

For a convex body $K$ the assignment $\|x\|_K := \inf\{\lambda > 0 : \lambda x \in K\}$ defines an *(asymmetric) norm* on $V$ with unit ball $K$. If $K$ is centrally-symmetric with respect to the origin, then $\|\cdot\|_K$ is an actual norm. In that case the polar body $K^\circ$ is also centrally-symmetric and $\|\cdot\|_{K^\circ}$ is the *dual norm.* Norms and support functions are related by

$$\|\ell\|_{K^\circ} \;=\; h(K;\ell) \qquad \text{for all } \ell \in V^*.$$

In particular, if $K$ is an orbitope, then $\|\cdot\|_K$ and $\|\cdot\|_{K^\circ}$ are $G$-equivariant norms.

Every point $p$ in a convex body $K$ is in the convex hull of finitely many extreme points. We denote by $d_p$ the least cardinality of a set $E$ of extreme points with $p \in \operatorname{conv}(E)$. We call $\mathfrak{c}(K) := \sup\{d_p : p \in K\}$ the *Carathéodory number* of $K$. Carathéodory's Theorem (see e.g. [3, §I.2]) asserts that $\mathfrak{c}(K)$ is bounded from above by $\dim K + 1$. Fenchel showed that $\mathfrak{c}(K) \leq \dim K$ when the set of extreme points of $K$ is connected [12]. Note that the Carathéodory number of an orbitope $\mathcal{O}_v = \operatorname{conv}(G \cdot v)$ in general depends on $v$ (cf. [23, Theorem 4.9]) whereas, for multiplicity-free representations, the dimension of $\mathcal{O}_v$ does not.

**Question 3.** *What are the Carathéodory numbers of orbitopes?*

We refer to the recent work of Barvinok and Blekherman [4, 7] for more information about the convex geometry of orbitopes and coorbitopes, especially with regard to their volumes.

2.2. **Algebraic geometry.** Here we look at orbitopes as objects in real algebraic geometry. Fix a rational representation $\rho : G \to GL(m, \mathbb{R})$ of a compact connected algebraic group $G$. Every orbit $G \cdot v$ is an irreducible real algebraic variety in $\mathbb{R}^m$, and we may ask for its prime ideal. By the Tarski-Seidenberg Theorem [6, §2.4], the orbitope is a semi-algebraic set.

**Question 4.** *Which orbitopes are* basic *semi-algebraic sets, i.e. for which triples $(G, \rho, v)$ can* $\operatorname{conv}(G \cdot v)$ *be described by a finite conjunction of polynomial equations and inequalities?*

The boundary $\partial\mathcal{O}$ of an orbitope $\mathcal{O}$ in $\mathbb{R}^m$ is a compact semi-algebraic set of codimension one in its affine span $\operatorname{aff}(\mathcal{O})$. The Zariski closure of $\partial\mathcal{O}$ is denoted by $\partial_a\mathcal{O}$. We call it the *algebraic boundary* of $\mathcal{O}$. If $\operatorname{aff}(\mathcal{O}) = \mathbb{R}^m$ then the algebraic boundary $\partial_a\mathcal{O}$ is the zero set of a unique (up to scaling) reduced polynomial $f(x_1, \ldots, x_m)$ whose coefficients lie in the field of definition of $(G, \rho, v)$. That field of definition will often be the rational numbers $\mathbb{Q}$. Since scalars in $\mathbb{Q}$ have an exact representation in computer algebra, but scalars in $\mathbb{R}$ require numerical floating point approximations, we seek to use $\mathbb{Q}$ instead of $\mathbb{R}$ wherever possible.

**Question 5.** *How to calculate the algebraic boundary $\partial_a\mathcal{O}$ of an orbitope $\mathcal{O}$?*

The irreducible factors of the polynomial $f(x_1, \ldots, x_m)$ that cuts out $\partial_a\mathcal{O}$ arise from various singularities in the boundary $\partial\mathcal{O}^o$ of the coorbitope $\mathcal{O}^o$. We believe that a complete answer to Question 5 will involve a Whitney stratification of the real algebraic hypersurface

$\partial_a \mathcal{O}^o$. Recall that a *Whitney stratification* is a decomposition into locally closed submanifolds (strata) in which the singularity type of each stratum is locally constant along the stratum. The faces of a polytope form a Whitney stratification of its boundary, which is dual to the stratification of the polar polytope. We expect a similar duality between the Whitney stratification of the boundary of an orbitope and of the boundary of its coorbitope.

**Question 6.** *How to compute and study the algebraic boundary $\partial_a \mathcal{O}^o$ of the coorbitope $\mathcal{O}^o$? Is every component of $\partial_a \mathcal{O}$ the dual variety to a stratum in a Whitney stratification of $\partial_a \mathcal{O}^o$?*

Recall that the *dual variety* $X^\vee$ of a subvariety $X$ in $\mathbb{R}^m$ is the Zariski closure of the set of all affine hyperplanes that are tangent to $X$ at some regular point. When studying this duality, algebraic geometers usually work in complex projective space $\mathbb{P}^m_\mathbb{C}$ rather than real affine space $\mathbb{R}^m$. In some of the examples for $G = SO(n)$ seen in this paper, the algebraic boundary $\partial_a \mathcal{O}^o$ of the coorbitope $\mathcal{O}^o$ coincides with the dual variety $X^\vee$ of the orbit $X = G{\cdot}v$. A good example for this is the discriminantal hypersurface in Corollary 6.4. More generally, we have the impression that the hypersurface $\partial_a \mathcal{O}^o$ is often irreducible while $\partial_a \mathcal{O}$ tends to be reducible. For further appearances of dual varieties in convex algebraic geometry see [27, 35].

The *k-th secant variety* of $G \cdot v$ is the Zariski closure of all $(k{+}1)$-flats spanned by points of $G \cdot v$. The study of secant varieties leads to lower bounds for the Carathéodory number:

**Proposition 2.3.** *If $k \geq \mathfrak{c}(\mathcal{O}_v)$ then the $k$-th secant variety of $G \cdot v$ is the ambient space $\mathbb{R}^m$.*

*Proof.* Let $k \geq \mathfrak{c}(\mathcal{O}_v)$. The set of points that lie in the convex hull of $k + 1$ points of $G \cdot v$ is dense in $\mathcal{O}_v$ and hence is Zariski dense in $\mathbb{R}^m$. The $k$-th secant variety contains that set. □

The lower bound for $\mathfrak{c}(\mathcal{O}_v)$ from Proposition 2.3 usually does not match Fenchel's upper bound $\mathfrak{c}(\mathcal{O}_v) \leq \dim(\mathcal{O}_v)$. For instance, consider the Carathéodory orbitope $\mathcal{O}_v$ in Example 1.1. Its algebraic boundary $\partial_a \mathcal{O}_v$ equals the second secant variety of the orbit $G \cdot v$, so Proposition 2.3 implies $\mathfrak{c}(\mathcal{O}_v) \geq 3$. This orbitope satisfies $\mathfrak{c}(\mathcal{O}_v) = 3$ but $\dim(\mathcal{O}_v) = 4$.

**Question 7.** *For which orbitopes $\mathcal{O}$ is the boundary $\partial_a \mathcal{O}$ one of the secant varieties of $G{\cdot}v$ ? When is the lower bound on the Carathéodory number $\mathfrak{c}(\mathcal{O})$ in Proposition 2.3 tight ?*

2.3. **Optimization.** A fundamental object in convex optimization is the set $\mathrm{PSD}_n$ of positive semidefinite symmetric real $n{\times}n$-matrices. This is the closed basic semi-algebraic cone defined by the non-negativity of the $2^n - 1$ principal minors. It can also be described by only $n$ polynomial inequalities, namely, by requiring that the elementary symmetric polynomials in the eigenvalues, i.e. the suitably normalized coefficients of the characteristic polynomial, be non-negative. The algebraic boundary $\partial_a \mathrm{PSD}_n$ of the cone $\mathrm{PSD}_n$ is the symmetric $n{\times}n$-determinant. All faces of $\mathrm{PSD}_n$ are exposed, isomorphic to $\mathrm{PSD}_k$ for $k \leq n$, and indexed by the lattice of linear subspaces ordered by reverse inclusion.

Spectrahedra inherit these favorable properties. Recall that a *spectrahedron* is the intersection of the cone $\mathrm{PSD}_n$ with an affine-linear subspace in $\mathrm{Sym}_2(\mathbb{R}^n)$. If we know that an orbitope is a spectrahedron then this either answers or simplifies many of our questions.

**Question 8.** *Characterize all* $\mathrm{SO}(n)$*-orbitopes that are spectrahedra.*

Polytopes are special cases of spectrahedra: they arise when the affine-linear space consists of diagonal matrices. One major distinction between polytopes and spectrahedra is that the class of spectrahedra is not closed under projection. That is, the image of a spectrahedron under a linear map is typically not a spectrahedron. See Section 5 for orbitopal examples. Characterizing projections of spectrahedra among all convex bodies is a major open problem in optimization theory; see e.g. [18]. Here is a special case of this general problem:

**Question 9.** *Is every orbitope the linear projection of a spectrahedron?*

In Questions 8 and 9, it is important to keep track of the subfield of $\mathbb{R}$ over which the data $(G, \rho, v)$ are defined. Frequently, this subfield is the rational numbers $\mathbb{Q}$, and in this case we seek to write the orbitope as a (projected) spectrahedron over $\mathbb{Q}$ and not just over $\mathbb{R}$.

Semidefinite programming is the problem of maximizing a linear function over a (projected) spectrahedron, and there are efficient numerical algorithms for solving this problem in practice. In our context of orbitopes, the optimization problem can be phrased as follows:

**Question 10.** *What method can be used for maximizing a linear function $\ell$ over an orbitope $\mathcal{O}$? Equivalently, how to evaluate the norm $\ell \mapsto \|\ell\|_{\mathcal{O}^\circ}$ associated with the coorbitope $\mathcal{O}^\circ$ ?*

This is equivalent to a non-linear optimization problem over the compact group $G$. We seek to find $g \in G$ which maximizes $\ell(\rho(g) \cdot v)$. This maximum is an algebraic function of $v$.

## 3. Schur-Horn Orbitopes

In this section we study two families of orbitopes for the orthogonal group $G = O(n)$. This group acts on the Lie algebra $\mathfrak{gl}_n$ by restricting the adjoint representation of $GL(n, \mathbb{R})$. The $O(n)$-module $\mathfrak{gl}_n$ decomposes into two distinguished invariant subspaces, namely $\mathrm{Sym}_2\mathbb{R}^n$ and $\wedge_2\mathbb{R}^n$. These correspond to the normal and tangent space of $O(n) \subset GL(n, \mathbb{R})$ at the identity. In matrix terms, the spaces of symmetric and skew-symmetric matrices form two natural representations of $O(n)$ for the action $g \cdot A = gAg^T$ with $g \in O(n)$ and $A \in \mathbb{R}^{n\times n}$.

For a symmetric matrix $M \in \mathrm{Sym}_2\mathbb{R}^n$ we define the *symmetric Schur-Horn orbitope*

$$\mathcal{O}_M \;:=\; \mathrm{conv}(G \cdot M) \;\subset\; \mathrm{Sym}_2\mathbb{R}^n.$$

For a skew-symmetric matrix $N \in \wedge_2\mathbb{R}^n$ we define the *skew-symmetric Schur-Horn orbitope*

$$\mathcal{O}_N \;:=\; \mathrm{conv}(G \cdot N) \subset \wedge_2\mathbb{R}^n.$$

We shall see that these orbitopes are intimately related to certain polytopes which govern their boundary structure and spectrahedral representation. This connection arises via the classical Schur-Horn theorem [33]. The material in the sections below could also be presented in symplectic language, using the moment maps of Atiyah-Guillemin-Sternberg [1, 15].

### 3.1. **Symmetric Schur-Horn orbitopes.**

The $\binom{n+1}{2}$-dimensional space $\mathrm{Sym}_2\mathbb{R}^n$ decomposes into the trivial $O(n)$-representation, given by multiples of the identity matrix, and the irreducible representation of symmetric $n\times n$-matrices with trace zero. Every symmetric matrix $M \in \mathrm{Sym}_2\mathbb{R}^n$ is orthogonally diagonalizable over $\mathbb{R}$. The ordered list of eigenvalues of $M$ is denoted $\lambda(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M))$. The orbit $G \cdot M$ equals the set of matrices $A \in \mathrm{Sym}_2\mathbb{R}^n$ that satisfy $\lambda(A) = \lambda(M)$. We shall see that its convex hull $\mathcal{O}_M = \mathrm{conv}(G \cdot M)$ is the set of matrices $A \in \mathrm{Sym}_2\mathbb{R}^n$ for which $\lambda(A)$ is majorized by $\lambda(M)$.

For $p, q \in \mathbb{R}^n$ we say that $q$ is *majorized* by $p$, written $q \trianglelefteq p$, if $q_1 + q_2 + \cdots + q_n = p_1 + p_2 + \cdots + p_n$, and, after reordering, $q_1 \geq \cdots \geq q_n$ and $p_1 \geq \cdots \geq p_n$, we have

$$q_1 + q_2 + \cdots + q_k \;\leq\; p_1 + p_2 + \cdots + p_k \quad \text{for } k = 1, \ldots, n-1.$$

For a fixed point $p \in \mathbb{R}^n$, the set of all points $q$ majorized by $p$ is the convex polytope

$$\Pi(p) \;=\; \{q \in \mathbb{R}^n : q \trianglelefteq p\} \;=\; \mathrm{conv}\{\pi \cdot p = (p_{\pi(1)}, \ldots, p_{\pi(n)}) : \pi \in \mathfrak{S}_n\}.$$

Here $\mathfrak{S}_n$ denotes the symmetric group, and $\Pi(p)$ is the *permutahedron* with respect to $p$. This is a well-studied polytope [29, 38] and is itself an orbitope for $\mathfrak{S}_n$. The permutahedron $\Pi(p)$ for $p = (p_1 \geq p_2 \geq \cdots \geq p_n)$ consists of all points $q \in \mathbb{R}^n$ such that $\sum_i p_i = \sum_i q_i$ and

$$\sum_{i \in I} q_i \;\leq\; \sum_{i=1}^{|I|} p_i \qquad \text{for all } I \subseteq [n]. \tag{3.1}$$

Let $\mathsf{D} : \mathrm{Sym}_2\mathbb{R}^n \to \mathbb{R}^n$ be the linear projection onto the diagonal.

**Proposition 3.1** (The symmetric Schur-Horn Theorem [22])**.** *Let $M \in \mathrm{Sym}_2\mathbb{R}^n$ and $\mathcal{O}_M$ its symmetric Schur-Horn orbitope. Then the diagonal $\mathsf{D}(M)$ is majorized by the vector of eigenvalues $\lambda(M)$. In fact, the orbitope $\mathcal{O}_M$ maps linearly onto the permutahedron:*

$$\mathsf{D}(\mathcal{O}_M) \;=\; \Pi(\lambda(M)).$$

**Corollary 3.2.** *The Schur-Horn orbitope equals $\mathcal{O}_M = \{A \in \mathrm{Sym}_2\mathbb{R}^n : \lambda(A) \trianglelefteq \lambda(M)\}$.*

*Proof.* We have shown that the right hand side equals $\{A \in \mathrm{Sym}_2\mathbb{R}^n : \lambda(A) \in \mathsf{D}(\mathcal{O}_M)\}$. We claim that a matrix $A$ is in this set if and only if $A$ lies in $\mathcal{O}_M$. This is clear if $A$ is a diagonal matrix. It follows for all matrices since both sets are invariant under the $O(n)$-action. ☐

Our next goal is to derive a spectrahedral characterization of $\mathcal{O}_M$. Consider the natural action of the Lie group $GL(n, \mathbb{R})$ on the $k$-th exterior power $\wedge_k\mathbb{R}^n$. If $\{v_1, v_2, \ldots, v_n\}$ is any basis of $\mathbb{R}^n$, then the induced basis vectors of $\wedge_k\mathbb{R}^n$ are $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. A matrix $g \in GL(n, \mathbb{R})$ acts on a basis element by sending it to $g \cdot v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} = (g \cdot v_{i_1}) \wedge (g \cdot v_{i_2}) \wedge \cdots \wedge (g \cdot v_{i_k})$. We denote by $\mathcal{L}_k : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\wedge_k\mathbb{R}^n)$ the induced map of Lie algebras. The linear map $\mathcal{L}_k$ is defined by the rule

$$\mathcal{L}_k(B)(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}) \;=\; \sum_{j=1}^{k} v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} \wedge (Bv_{i_j}) \wedge v_{i_{j+1}} \wedge \cdots \wedge v_{i_k}. \tag{3.2}$$

The $\binom{n}{k} \times \binom{n}{k}$-matrix that represents $\mathcal{L}_k(B)$ in the standard basis of $\wedge_k\mathbb{R}^n$ is known as the *$k$-th additive compound matrix* of the $n \times n$-matrix $B$. It has the following main property:

**Lemma 3.3.** *Let $B \in \mathrm{Sym}_2\mathbb{R}^n$ with eigenvalues $\lambda(B) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $\mathcal{L}_k(B)$ is symmetric and has eigenvalues $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.*

*Proof.* Let $v_1, \ldots, v_n$ be an eigenbasis for $B$. Then the formula (3.2) says

$$\mathcal{L}_k(B)(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}) \;=\; \sum_{j=1}^{k} v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} \wedge (\lambda_{i_j} v_{i_j}) \wedge v_{i_{j+1}} \wedge \cdots \wedge v_{i_k}.$$

Hence $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ is an eigenvector of $\mathcal{L}_k(B)$ with eigenvalue $\lambda_{i_1} + \cdots + \lambda_{i_k}$. □

This leads to the result that each symmetric Schur-Horn orbitope $\mathcal{O}_M$ is a spectrahedron.

**Theorem 3.4.** *Let $M \in \mathrm{Sym}_2\mathbb{R}^n$ with ordered eigenvalues $\lambda(M) = (\lambda_1 \geq \cdots \geq \lambda_n)$. Then*

$$\mathcal{O}_M \;=\; \Big\{ A \in \mathrm{Sym}_2\mathbb{R}^n \;:\; \mathrm{Tr}(A) = \mathrm{Tr}(M) \text{ and } \sum_{i=1}^{k} \lambda_i \mathrm{Id}_{\binom{n}{k}} - \mathcal{L}_k(A) \;\succeq\; 0 \text{ for } k = 1, \ldots, n-1 \Big\}.$$

*Proof.* A matrix $A \in \mathrm{Sym}_2\mathbb{R}^n$ is in $\mathcal{O}_M$ if and only if $\lambda(A)$ is in the permutahedron $\Pi(\lambda(M))$. From the inequality representation in (3.1), in conjunction with Lemma 3.3, we see that this is the case if and only if the largest eigenvalue of $\lambda(\mathcal{L}_k(A))$ is at most $\lambda_1 + \cdots + \lambda_k$. □

We shall now derive the description of all faces of the Schur-Horn orbitope $\mathcal{O}_M$. Since $\mathcal{O}_M$ is a spectrahedron, by Theorem 3.4, we know that all of its faces are exposed faces. Hence a face of $\mathcal{O}_M$ is the set of points maximizing a linear function $\ell : \mathrm{Sym}_2\mathbb{R}^n \to \mathbb{R}$. The canonical $O(n)$-invariant inner product on $\mathrm{Sym}_2\mathbb{R}^n$ is given by $\langle A, B\rangle = \mathrm{Tr}(AB)$ and, by identifying spaces, a linear function may be written as $\ell(\,\cdot\,) = \langle B, \cdot\,\rangle$. Note that the dual space $(\mathrm{Sym}_2\mathbb{R}^n)^*$ is equipped with the contragredient action, that is, $g \cdot \ell = \langle g^T B g, \cdot\,\rangle$.

**Theorem 3.5.** *Every $O(n)$-orbit of faces of $\mathcal{O}_M$ intersects the pullback of a unique $\mathfrak{S}_n$-orbit of faces of the permutahedron $\Pi(\lambda(M))$. In particular, the faces of $\mathcal{O}_M$ are products of symmetric Schur-Horn orbitopes of smaller dimensions corresponding to flags in $\mathbb{R}^n$.*

*Proof.* Let $F$ be a face of $\mathcal{O}_M$ and let $\ell = \langle B, \cdot\,\rangle$ be a linear function maximized at $F$. Then the face $g \cdot F$ is given by $g \cdot \ell$ and we may identify $O(n) \cdot F$ with the orbit $O(n) \cdot \ell$. Since $B$ is orthogonally diagonalizable, the orbit $O(n) \cdot \ell$ contains the diagonal matrices $\pi \cdot \lambda(B)$ for $\pi \in \mathfrak{S}_n$. The corresponding faces are the pullbacks of the orbit of faces of $\Pi(\lambda(M))$ given by the linear function $\langle \lambda(B), \cdot \rangle_{\mathbb{R}^n}$. Faces of the permutahedron correspond to flags of coordinate subspaces, and $O(n)$ translates these to arbitrary flags of subspaces. □

Facets of the permutahedron $\Pi(p)$ correspond to coordinate subspaces. Replacing these with arbitrary subspaces yields supporting hyperplanes for the Schur-Horn orbitope $\mathcal{O}_M$.

**Corollary 3.6.** *The Schur-Horn orbitope $\mathcal{O}_M$ is the set of matrices $A \in \mathrm{Sym}_2\mathbb{R}^n$ such that*

$$\mathrm{Tr}(A|_L) \leq \mathrm{Tr}(M|_L) \quad \text{for every subspace } L \subseteq \mathbb{R}^n.$$

The following three examples show how Theorem 3.5 translates into explicit face lattices.

**Example 3.7** (The free spectrahedron)**.** Let $M = e_1 e_1^T \in \mathrm{Sym}_2\mathbb{R}^n$ be the diagonal matrix with diagonal $(1, 0, \ldots, 0)$. The orbitope $\mathcal{O}_M$ is the convex hull of all symmetric rank 1 matrices with trace 1, and hence $\mathcal{O}_M = \mathrm{PSD}_n \cap \{\mathrm{trace} = 1\}$. This orbitope plays the role of a "simplex among spectrahedra" because every compact spectrahedron is an affine section.

The face $\mathcal{O}_M^B$ of the orbitope $\mathcal{O}_M$ in direction $B \in \mathrm{Sym}_2\mathbb{R}^n$ is isomorphic to

$$\mathrm{conv}\{uu^T : u \in \mathbb{S}^{n-1} \cap \mathrm{Eig}_{\max}(B)\}$$

where $\mathbb{S}^{n-1}$ is the unit sphere and $\mathrm{Eig}_{\max}(B)$ is the eigenspace of $B$ with maximal eigenvalue. Thus, $\mathcal{O}_M^B$ is isomorphic to a lower dimensional Schur-Horn orbitope for a rank one matrix.

We conclude that the face lattice of $\mathcal{O}_M$ consists of the linear subspaces of $\mathbb{R}^n$ ordered by inclusion. This fact is well known; see [3, §II.12]. The dimension of a face corresponding to a $k$-subspace is $\binom{k+1}{2} - 1$. The projection $\mathsf{D}(\mathcal{O}_M)$ is the standard $(n-1)$-dimensional simplex $\Delta_{n-1} = \mathrm{conv}\{e_1, e_2, \ldots, e_n\}$ whose faces correspond to the coordinate subspaces. □

**Example 3.8** (Spectrahedral hypersimplices)**.** We now describe continuous analogs to hypersimplices, extending the simplices in Example 3.7. Fix $0 < k < n$ and let $M \in \mathrm{Sym}_2\mathbb{R}^n$ be the diagonal matrix with $k$ ones and $n-k$ zeros. The orbitope $\mathcal{O}_M$ of the $(n,k)$-*spectrahedral hypersimplex*, as its diagonal projection $\mathsf{D}(\mathcal{O}_M) = \Delta(n,k)$ is the classical $(n,k)$-hypersimplex; cf. [38, Example 0.11]. For instance, if $n = 4$ and $k = 2$ then $\mathcal{O}_M$ is nine-dimensional and $\mathsf{D}(\mathcal{O}_M)$ is an octahedron. Up to $\mathfrak{S}_4$-symmetry, the octahedron has one orbit of vertices and edges but two orbits of triangles. The pullback of any edge is a circle, and the pullbacks of the triangles are five-dimensional symmetric Schur-Horn orbitopes $\mathcal{O}_M$ for $\lambda(M) = (1,0,0)$ and $\lambda(M) = (1,1,0)$. Both facets are isomorphic to free spectrahedra. □

**Example 3.9** (The generic symmetric Schur-Horn orbitope)**.** Let $M \in \mathrm{Sym}_2\mathbb{R}^n$ be a symmetric matrix with distinct eigenvalues, e.g. $\lambda(M) = (1,2,\ldots,n)$. The image of $\mathcal{O}_M$ under the diagonal map is the classical permutahedron $\Pi_n = \Pi(1,2,3,\ldots,n)$. Its face lattice may be described as the collection of all flags of coordinate subspaces in $\mathbb{R}^n$ ordered by refinement.

We may associate to every $B \in \mathrm{Sym}_2\mathbb{R}^n$ the complete flag whose $k$-th subspace is the direct sum of the eigenspaces of the first $k$ largest eigenvalues. Thus, $O(n) \cdot M$ may be identified with the complete flag variety over $\mathbb{R}$. As for the facial structure, the face $\mathcal{O}_M^B$ is isomorphic to the convex hull of the orbit $\mathrm{stab}_{O(n)}(B) \cdot M$. Here, the stabilizer decomposes into a product of groups isomorphic to $O(d_i)$ where $d_i$ is the dimension of the $i$-th eigenspace of $B$. Hence, the face $\mathcal{O}_M^B$ is isomorphic to a Cartesian product of generic Schur-Horn orbitopes and is of dimension $\sum_i \binom{d_i+1}{2}$. The face only depends on the flag associated to $B$. This implies that the face lattice of $\mathcal{O}_M$ is isomorphic to the set of partial flags ordered by refinement. Again, in every orbit of flags there is a flag consisting only of coordinate subspaces. These special flags form the face lattice of the standard permutahedron $\Pi(1,2,\ldots,d) = \mathsf{D}(\mathcal{O}_M)$. We regard $\mathcal{O}_M$ as a continuous analog of the permutahedron. □

We conclude this subsection with a discussion of the algebraic boundary $\partial_a\mathcal{O}_M$ of the Schur-Horn orbitope. Let $\mathbb{K}$ be the smallest subfield of $\mathbb{R}$ that contains the eigenvalues $\lambda_1, \ldots, \lambda_n$, and suppose that the $\lambda_i$ are sufficiently general. Then the hypersurface $\partial_a\mathcal{O}_M$ is defined in the affine space $\{A \in \mathrm{Sym}_2(\mathbb{R}^n) \mid \mathrm{Tr}(A) = \mathrm{Tr}(M)\}$ by the following polynomial of degree $2^n - 2$ in $\binom{n+1}{2}$ unknowns over the field $\mathbb{K}$:

$$f(A) \quad = \quad \prod_{k=1}^{n-1} \det\Bigl(\mathcal{L}_k(A) - \sum_{i=1}^{k} \lambda_i \cdot \mathrm{Id}_{\binom{n}{k}}\Bigr).$$

However, from a computer algebra perspective, this is not what we want. Assuming that $M$ has entries in $\mathbb{Q}$, we prefer not to pass to the field extension $\mathbb{K}$, but we want the algebraic boundary $\partial_a \mathcal{O}_M$ to be the $\mathbb{Q}$-Zariski closure of the above hypersurface $\{f(A) = 0\}$. For instance, suppose that the characteristic polynomial of $M$ is irreducible over $\mathbb{Q}$. Then we must take the product of $f(A)$ over all permutations of the eigenvalues $\lambda_1, \ldots, \lambda_n$, and the polynomial $g(A)$ that defines $\partial_a \mathcal{O}_M$ over $\mathbb{Q}$ is the reduced part of that product. It equals

$$g(A) \quad = \quad \prod_{k=1}^{\lceil n/2 \rceil} \det\bigl(\mathcal{L}_k(A) \oplus \mathcal{L}_k(-M)\bigr),$$

where $\oplus$ denotes the *tensor sum* of two square matrices of the same size (see e.g. [28, §3]). Here the product goes only up to $\lceil n/2 \rceil$ because the matrices $A$ and $M$ have the same trace.

For special matrices $M$, the characteristic polynomial may factor over $\mathbb{Q}$, and in this case the algebraic boundary $\partial_a \mathcal{O}_M$ is cut out by a factor of the polynomial $f(A)$ or $g(A)$.

3.2. **Skew-symmetric Schur-Horn orbitopes.** The space $\wedge_2 \mathbb{R}^n$ consists of skew-symmetric $n \times n$-matrices $N$. The eigenvalues of $N$ are purely imaginary, say $\pm i\tilde{\lambda}_1, \ldots, \pm i\tilde{\lambda}_k$, where $i = \sqrt{-1}$, with $k = \lfloor \frac{n}{2} \rfloor$ and an additional 0 eigenvalue if $n$ is odd. Thus $N$ is not diagonalizable over $\mathbb{R}$, but the adjoint $O(n)$-action brings the matrix $N$ into the normal form

$$gNg^T \; = \; \begin{pmatrix} & \Lambda \\ -\Lambda & \end{pmatrix} \text{ for } n \text{ even} \quad \text{and} \quad gNg^T \; = \; \begin{pmatrix} & & \Lambda \\ & 0 & \\ -\Lambda & & \end{pmatrix} \text{ for } n \text{ odd.}$$

Here $g$ is a suitable matrix in $O(n)$, $\Lambda$ is the diagonal matrix with diagonal $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k \geq 0$, and we denote $\tilde{\lambda}(N) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k)$. Let $\mathsf{SD} : \wedge_2 \mathbb{R}^n \to \mathbb{R}^k$ be the linear map such that

$$\begin{aligned} \mathsf{SD}(N) \; &= \; (N_{1,k} \quad, N_{2,k+1}, \ldots, N_{k,n}) \qquad \text{if } n = 2k, \text{ and} \\ \mathsf{SD}(N) \; &= \; (N_{1,k+1}, N_{2,k+2}, \ldots, N_{k,n}) \qquad \text{if } n = 2k+1. \end{aligned}$$

If $N$ is in normal form as above, then $\mathsf{SD}(N) = \mathsf{D}(\Lambda) = \tilde{\lambda}(N)$. We call $\mathsf{SD}(N)$ the *skew-diagonal* of $N$. In analogy to the symmetric case, the set $\mathsf{SD}(\mathcal{O}_N)$ of all skew-diagonals arising from $\mathcal{O}_M$ is nicely behaved; in fact, the necessary changes are rather modest.

For a point $q \in \mathbb{R}^k$ we denote by $|q| = (|q_1|, \ldots, |q_k|)$ the vector of absolute values. For $p \in \mathbb{R}^k$ let $\Pi^s(p)$ be the set of points $q \in \mathbb{R}^k$ such that $|q|$ is *weakly majorized* by $|p|$. This means that $|p|$ and $|q|$ satisfy the majorization conditions except that $\sum_i |q_i| \leq \sum_i |p_i|$ is allowed. The polytope $\Pi^s(P)$ is the *$B_k$-permutahedron.* It is the convex hull of the orbit of $p$ under the action of the Coxeter group $B_k$, the group of all $2^k \cdot k!$ signed permutations. The

$B_k$-permutahedron $\Pi^s(p)$ for $p = (p_1 \geq p_2 \geq \cdots \geq p_k)$ consists of all points $q \in \mathbb{R}^k$ with

$$\sum_{i\in I} q_i - \sum_{j\in J} q_j \;\leq\; \sum_{i=1}^{|I\cup J|} p_i \quad \text{for any } I, J \subseteq [k] \text{ with } I \cap J = \emptyset. \tag{3.3}$$

As expected, we have the following analog of the symmetric Schur-Horn theorem.

**Proposition 3.10** (The skew-symmetric Schur-Horn theorem [22])**.** *Let $N \in \wedge_2\mathbb{R}^n$ with $\tilde\lambda(N) = (\tilde\lambda_1 \geq \cdots \geq \tilde\lambda_k)$ and $\mathcal{O}_N$ the skew-symmetric Schur-Horn orbitope of $N$. Then $|\mathsf{SD}(N)|$ is weakly majorized by $(\tilde\lambda_1, \ldots, \tilde\lambda_k)$. In particular, we have $\mathsf{SD}(\mathcal{O}_N) = \Pi^s(\tilde\lambda)$.*

The same arguments as in the symmetric case yield the following results.

**Theorem 3.11.** *Every $O(n)$ orbit of faces of the skew-symmetric Schur-Horn orbitope $\mathcal{O}_N$ contains the pullback of a unique $B_k$-orbit of faces of the $B_k$-permutahedron $\Pi^s(\tilde\lambda(N))$.*

**Corollary 3.12.** *The skew-symmetric Schur-Horn orbitope $\mathcal{O}_N$ coincides with the set of skew-symmetric matrices $A$ such that $|\mathsf{SD}(A)|$ is weakly majorized by $|\mathsf{SD}(N)|$.*

**Example 3.13.** Fix $n = 6$ and $k = 3$, and $p = (1, 2, 3)$. Then the system (3.3) consists of 26 linear inequalities, namely, six inequalities $\pm q_i \leq 3$, twelve inequalities $\pm q_i \pm q_j \leq 5$, and eight inequalities $\pm q_i \pm q_j \pm q_k \leq 6$. Their solution set is the $B_3$-permutahedron $\Pi^s(1, 2, 3)$, commonly known as the *truncated cuboctahedron*, and it has 48 vertices, 72 edges and 26 facets (six octagons, twelve squares and eight hexagons). A picture is shown in Figure 2.

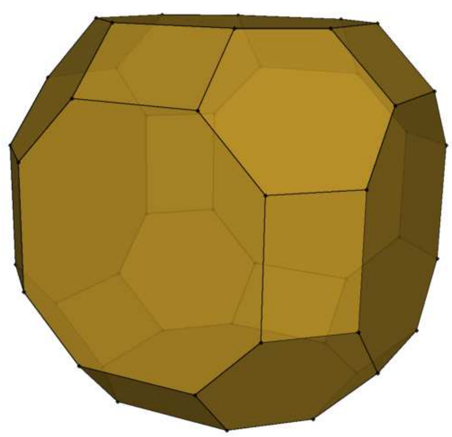


FIGURE 2. The $B_3$-permutahedron is the truncated cuboctahedron.

Let $N$ denote a general skew-symmetric 6×6-matrix. Theorem 3.11 implies that the facets of the 15-dimensional orbitope $\mathcal{O}_N$ come in three families, corresponding to $O(6)$-orbits of the octagons, squares, and hexagons in Figure 2. The facets arising from the octagons are skew-symmetric Schur-Horn orbitopes for $SO(4)$ with skew-diagonal $(1, 2)$ and therefore have dimension six. The facets arising from the squares are the product of a line segment and a disc, coming from $O(2) \times SO(2)$ with $O(2)$ acting by the determinant. The facets arising from the hexagons are $O(3)$-orbitopes isomorphic to symmetric Schur-Horn orbitopes with eigenvalues $(1,2,3)$ and therefore have dimension five. □

We next present a spectrahedral description of an arbitrary skew-symmetric Schur-Horn orbitope $\mathcal{O}_N$. To derive this, we return to symmetric matrices and their real eigenvalues.

**Lemma 3.14.** *Let $N \in \wedge_2\mathbb{R}^n$ be a matrix with eigenvalues $\pm i\tilde{\lambda}_1, \ldots, \pm i\tilde{\lambda}_k$ and let*

$$\hat{N} \;=\; \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix} \;\in\; \mathrm{Sym}_2\mathbb{R}^{2n}.$$

*Then $\hat{N}$ has eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k, \tilde{\lambda}_k, -\tilde{\lambda}_1, -\tilde{\lambda}_1, -\tilde{\lambda}_2, -\tilde{\lambda}_2, \ldots, -\tilde{\lambda}_k, -\tilde{\lambda}_k$. For any $1 \le j \le k$, the additive compound matrix $\mathcal{L}_{2j}(\hat{N})$ has largest eigenvalue $2(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_j)$.*

We conclude that each skew-symmetric Skew-Horn orbitope $\Pi^s(\tilde{\lambda}(N))$ is a spectrahedron:

**Theorem 3.15.** *Let $N \in \wedge_2\mathbb{R}^n$ with $\tilde{\lambda}(N) = (\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k)$. Then*

$$\mathcal{O}_N \;=\; \left\{ A \in \wedge_2\mathbb{R}^n : 2(\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_j)\mathrm{Id}_{\binom{2n}{2j}} - \mathcal{L}_{2j}(\hat{A}) \;\succeq\; 0 \;\text{ for } j = 1, \ldots, k \right\}.$$

From this theorem we can derive a description of the algebraic boundary as before, and again the issue arises that the $\tilde{\lambda}_j$ lie in an extension $\mathbb{K}$ over the field of definition of $N$, which will usually be $\mathbb{Q}$. At present we do not know whether Theorems 3.4 and 3.15 can be extended to obtain spectrahedral representations of the respective orbitopes over $\mathbb{Q}$.

We close this section with one more example of a skew-symmetric Schur-Horn orbitope.

**Example 3.16.** Consider the skew-symmetric Schur-Horn orbitope $\mathcal{O}_N$ for some $N \in \wedge_2\mathbb{R}^n$ with $\tilde{\lambda}(N) = (1, 0, \ldots, 0) \in \mathbb{R}^k$. According to Theorem 3.15, a spectrahedral representation is

$$\mathcal{O}_N \;=\; \left\{ A \in \wedge_2\mathbb{R}^n : \mathcal{L}_2(\hat{A}) \;\preceq\; 2 \cdot \mathrm{Id}_{2n} \right\}.$$

The projection $\mathsf{SD}(\mathcal{O}_N)$ to the skew diagonal is the *crosspolytope* $\mathrm{conv}\{\pm e_1, \ldots, \pm e_k\}$. This a regular polytope with symmetry group $B_k$, and it has only one orbit of faces in each dimension. The orbitope $\mathcal{O}_N$ is the $d = 2$ instance of the *Grassmann orbitope* $\mathcal{G}_{d,n}$. These are important in the theory of calibrated manifolds, and we shall study them in Section 7. □

## 4. Tautological Orbitopes

We argued in Section 2 that, given a compact group $G$ acting algebraically on $V \simeq \mathbb{R}^n$, we can identify $G$ with a subgroup of $O(n)$, or even of $SO(n)$ when $G$ is connected. The ambient space $\mathrm{End}(V) \simeq \mathfrak{gl}_n$ is itself an $n^2$-dimensional real representation of the group $G$. The action of $G$ on $\mathrm{End}(V)$ is by left multiplication. The orbit of the identity matrix $\mathrm{Id}_n$ under this action is the group $G$ itself. We call the corresponding orbitope $\mathrm{conv}(G) = \mathrm{conv}(G \cdot \mathrm{Id}_n)$ the *tautological orbitope* for the pair $(G, V)$. This orbitope lives in $\mathrm{End}(V)$, and it serves as an initial object because it maps linearly to every orbitope $\mathrm{conv}(G \cdot v)$ in $V$. Tautological orbitopes of finite permutation groups have been studied under the name of *permutation* *polytopes* (see [29]). The most famous of them all is the *Birkhoff polytope* for $G = \mathfrak{S}_n$, which was studied for other Coxeter groups by McCarthy, Ogilvie, Zobin, and Zobin [25].

In this section we investigate the tautological orbitopes for the full groups $O(n)$ and $SO(n)$. Similar to the Schur-Horn orbitopes in Section 3, the facial structure is governed by polytopes arising from the projection onto the diagonal. We begin with the example $G = SO(3)$.

4.1. **Rotations in 3-dimensional space.** The group $SO(3)$ of $3\times 3$ rotation matrices has dimension three. Its tautological orbitope is a convex body of dimension nine. The following spectrahedral representation was suggested to us by Pablo Parrilo.

**Proposition 4.1.** *The tautological orbitope* $\operatorname{conv}(SO(3))$ *is a spectrahedron whose boundary is a quartic hypersurface. In fact, a* $3\times 3$*-matrix* $X = (x_{ij})$ *lies in* $\operatorname{conv}(SO(3))$ *if and only if*

$$\begin{pmatrix} 1{+}x_{11}{+}x_{22}{+}x_{33} & x_{32}-x_{23} & x_{13}-x_{31} & x_{21}-x_{12} \\ x_{32}-x_{23} & 1{+}x_{11}{-}x_{22}{-}x_{33} & x_{21}+x_{12} & x_{13}+x_{31} \\ x_{13}-x_{31} & x_{21}+x_{12} & 1{-}x_{11}{+}x_{22}{-}x_{33} & x_{32}+x_{23} \\ x_{21}-x_{12} & x_{13}+x_{31} & x_{32}+x_{23} & 1{-}x_{11}{-}x_{22}{+}x_{33} \end{pmatrix} \succeq 0. \tag{4.1}$$

*Proof.* We first claim that $\operatorname{conv}(SO(3))$ coincides with the set of all $3\times 3$-matrices

$$\begin{pmatrix} u_{11}{+}u_{22}{-}u_{33}{-}u_{44} & 2u_{23}-2u_{14} & 2u_{13}+2u_{24} \\ 2u_{23}+2u_{14} & u_{11}{-}u_{22}{+}u_{33}{-}u_{44} & 2u_{34}-2u_{12} \\ 2u_{24}-2u_{13} & 2u_{12}+2u_{34} & u_{11}{-}u_{22}{-}u_{33}{+}u_{44} \end{pmatrix} \tag{4.2}$$

where $U = (u_{ij})$ runs over all positive semidefinite $4\times 4$-matrices having trace 1.

Positive semidefinite $4\times 4$-matrices with both trace 1 and rank 1 are of the form

$$U \;=\; \frac{1}{a^2+b^2+c^2+d^2}\begin{pmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{pmatrix}.$$

Their convex hull is the free spectrahedron of Example 3.7. The image of the above rank 1 matrices $U$ under the linear map (4.2) is precisely the group $SO(3)$. This parametrization is known as the *Cayley transform*. Geometrically, it corresponds to the double cover $SU(2) \to SO(3)$. The claim follows because the linear map commutes with taking the convex hull.

The symmetric $4\times 4$-matrices $U = (u_{ij})$ with $\operatorname{trace}(U) = 1$ form a nine-dimensional affine space, and this space is isomorphic to the nine-dimensional space of all $3\times 3$-matrices $X = (x_{ij})$ under the linear map given in (4.2). We can express each $u_{ij}$ in terms of the $x_{kl}$ by inverting the linear relations $x_{11} = u_{11}+u_{22}-u_{33}-u_{44}$, $x_{12} = 2u_{23}-2u_{14}$, etc. The resulting symmetric $4\times 4$-matrix $U$ is precisely the matrix (4.1) in the statement of Proposition 4.1. □

The ideal of the group $O(3)$ is generated by the entries of the $3\times 3$-matrix $X\cdot X^T - \mathrm{Id}_3$, while the prime ideal of $SO(3)$ is that same ideal plus $\langle \det(X) - 1\rangle$. We can check that the prime ideal of $SO(3)$ coincides with the ideal generated by the $2\times 2$-minors of the matrix (4.1). Thus the group $SO(3)$ is recovered as the set of matrices (4.1) of rank one.

Proposition 4.1 implies that $\mathrm{conv}(SO(3))$ is affinely isomorphic to the free spectrahedron for $n = 4$, that is, to the set of positive semidefinite $4\times 4$-matrices with trace equal to 1. This implies a characterization of all faces of the tautological orbitope for $SO(3)$. First, all faces are exposed because $\mathrm{conv}(SO(3))$ is a spectrahedron. All of its proper faces are free spectrahedra, for $n = 1, 2, 3$, so they have dimensions 0, 2 and 5, as seen in Example 3.7.

4.2. **The orthogonal group.** We now examine $O(n) = \{X \in \mathbb{R}^{n\times n} : X \cdot X^T = \mathrm{Id}_n\}$. As before, let $\mathsf{D} : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ denote the projection of the $n\times n$-matrices onto their diagonals.

**Lemma 4.2.** *The projection* $\mathsf{D}(\mathrm{conv}(O(n))$ *of the tautological orbitope for the orthogonal group* $O(n)$ *to its diagonal is precisely the* $n$*-dimensional cube* $[-1, +1]^n$.

*Proof.* The columns of a matrix $X \in O(n)$ are unit vectors. Thus every coordinate $x_{ij}$ in bounded by 1 in absolute value and $\mathsf{D}(\mathrm{conv}(O(n))$ is a subset of the cube. For the reverse inclusion, note that all $2^n$ diagonal matrices with entries $\pm 1$ are orthogonal matrices. □

The cube $[-1, +1]^n$ is the special $B_n$-permutahedron $\Pi^s(1, 1, \ldots, 1)$. As with Schur-Horn orbitopes, the projection onto this polytope reveals the facial structure. As general endomorphisms are not normal, the key concept of diagonalizability is replaced by that of *singular value decomposition*. Recall that for any linear map $A \in \mathbb{R}^{n\times n}$ there are orthogonal transformations $U, V \in O(n)$ such that $UAV^T$ is diagonal with entries $\sigma(A) = (\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)) \in \mathbb{R}^n_{\geq 0}$. These entries are the *singular values* of $A$.

We shall see in (4.4) that $\mathrm{conv}(O(n))$ is a spectrahedron, hence all of its faces are exposed faces. The following result recursively characterizes all faces of this tautological orbitope.

**Theorem 4.3.** *Let* $\ell = \langle B, \cdot\rangle$ *be a linear function on* $\mathbb{R}^{n\times n}$ *with* $B \in \mathbb{R}^{n\times n}$*. Then the face of* $\mathrm{conv}(O(n))$ *in direction* $\ell$ *is isomorphic to* $\mathrm{conv}(O(m))$ *where* $m = \dim\ker(B)$.

*Proof.* Let $\ell(\cdot) = \langle B, \cdot\rangle$ be a linear function with $B \in \mathbb{R}^{n\times n}$, so that $\ell(A) = \mathrm{Trace}(AB)$. We fix a singular value decomposition $U\Sigma V = B$ of the matrix $B$. Here $\Sigma$ is a diagonal matrix $n\times n$ with its first $n - m$ entries positive and remaining $m$ entries zero. This matrix also defines a linear function $\ell'(\cdot) = \langle \Sigma, \cdot\rangle$ on $\mathbb{R}^{n\times n}$. Cyclic invariance of the trace ensures that the faces $\mathrm{conv}(O(n))^{\ell}$ and $\mathrm{conv}(O(n))^{\ell'}$ are isomorphic. The subset of $O(n)$ at which $\ell'$ is maximized is the subgroup $\{g \in O(n) : g \cdot e_i = e_i \text{ for } i = 1, \ldots, n - m\}$. The convex hull of this subgroup equals $\mathrm{conv}(O(n))^{\ell'}$. It coincides with the tautological orbitope for $O(m)$. □

We interpret Theorem 4.3 geometrically as saying that the tautological orbitope for $O(n)$ is a continuous analog of the $n$-dimensional cube. Every face of the cube is a smaller dimensional cube and the dimension of a face maximizing a linear functional $\ell$ is determined by the support of $\ell$. The role of the support is now played by the rank of the matrix $B$. This behavior yields information about the Carathéodory number of the tautological orbitope.

**Proposition 4.4.** *The Carathéodory number of the orbitope* $\mathrm{conv}(O(n))$ *is at most* $n + 1$.

*Proof.* By [23, Lemma 3.2], the Carathéodory number of a convex body $K$ is bounded via

$$\mathfrak{c}(K) \;\leq\; 1 + \max\{\mathfrak{c}(F) : F \subset K \text{ a proper face }\}.$$

Since every proper face is isomorphic to $\mathrm{conv}(O(k))$ for some $k < n$ the result follows by induction on $n$. The base case is $n = 1$ for which $\mathrm{conv}(O(1))$ is a 1-simplex. □

Note that the orbit $O(n) \cdot \mathrm{Id}_n$ coincides with the orbit of the identity matrix $\mathrm{Id}_n$ under the action of the product group $O(n) \times O(n)$ by both right and left multiplication. Hence the tautological orbitope $\mathrm{conv}(O(n))$ is also an $O(n) \times O(n)$-orbitope for that action. We shall now digress and study these orbitopes in general. After we have seen (in Theorem 4.7) that these are spectrahedra, we shall resume our discussion of $\mathrm{conv}(O(n))$.

4.3. **Fan orbitopes.** The group $G = O(n) \times O(n)$ acts on $\mathbb{R}^{n\times n}$ by simultaneous left and right translation. The action is given, for $(g, h) \in O(n) \times O(n)$ and $A \in \mathbb{R}^{n\times n}$, by

$$(g, h) \cdot A \ := \ gAh^T. \tag{4.3}$$

Ky Fan proved in [10] that the Schur-Horn theorem for symmetric matrices under conjugation by $O(n)$ generalizes to arbitrary square matrices under this $O(n)\times O(n)$ action. Now, singular values play the role of the eigenvalues. The following is a convex geometric reformulation:

**Lemma 4.5** (Ky Fan [10]). *For a square matrix $A \in \mathbb{R}^{n\times n}$ let $\mathcal{O}_A$ denote its orbitope under the action (4.3) of the group $O(n) \times O(n)$. Then the image $\mathsf{D}(\mathcal{O}_A)$ of the projection to the diagonal is the $B_n$-permutahedron with respect to the singular values $\sigma(A)$.*

We shall refer to $\mathcal{O}_A = \mathrm{conv}\{(g, h) \cdot A : g, h \in O(n)\}$ as the *Fan orbitope* of the matrix $A$. From Lemma 4.5, one easily deduces the analogous results to Theorems 3.5 and 3.11.

**Remark 4.6.** The facial structure of the Fan orbitope $\mathcal{O}_A$ is determined by the facial structure of the $B_n$-permutahedron $\Pi^s(\sigma(A))$ specified by the singular values of the matrix $A$.

The description of the $B_n$-permutahedron in terms of weak majorization was stated in (3.3). Rephrasing these same linear inequalities for the singular values, and using Lemma 4.5, now leads to a spectrahedral description of the Fan orbitopes. For that we make use of the alternative characterization of singular values as the square roots of the (non-negative) eigenvalues of $AA^T$. Using Schur complements, it can be seen that the $2n \times 2n$-matrix

$$S(A) \ = \ \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

has the eigenvalues $\pm\sigma_1(A), \ldots, \pm\sigma_n(A)$. We form its additive compound matrices as before.

**Theorem 4.7.** *Let $A$ be a real $n\times n$-matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Then its Fan orbitope $\mathcal{O}_A$ equals the spectrahedron*

$$\mathcal{O}_A \ = \ \left\{ X \in \mathbb{R}^{n\times n} \,:\, \mathcal{L}_k(S(X)) \preceq (\sigma_1 + \cdots + \sigma_k)\mathrm{Id}_{\binom{2n}{k}} \quad k = 1, \ldots, n \right\}.$$

Fix an integer $p \in \{1, 2, \ldots, n\}$. The *Ky Fan $p$-norm* is defined by

$$X \ \mapsto \ \sum_{i=1}^{p} \sigma_i(X).$$

This function is indeed a norm on $\mathbb{R}^{n\times n}$, and its unit ball is the Fan orbitope $\mathcal{O}_A$ where $A$ is the diagonal matrix with $p$ diagonal entries $1/p$ and $n-p$ diagonal entries 0. Theorem 4.7 shows that the unit ball in the Ky Fan $p$-norm is a spectrahedron. Two norms of special interest in applied linear algebra are the *operator norm* $\|A\| := \sigma_1(A)$ and the *nuclear norm* $\|A\|_* := \sigma_1(A) + \cdots + \sigma_n(A)$. Indeed, these two norms played the key role in work of Fazel, Recht and Parrilo [31] on compressed sensing in the matrix setting. Both the operator norm and the nuclear norm are closely tied to their vector counterparts.

**Remark 4.8.** The unit balls in the operator and nuclear norm are both Fan orbitopes. The projection to the diagonal yields the unit balls for the $\ell_\infty$ and the $\ell_1$-norm on $\mathbb{R}^n$. These two unit balls are polytopes in $\mathbb{R}^n$, namely, the $n$-cube and the $n$-crosspolytope respectively.

Fazel *et al.* showed in [31, Prop. 2.1] that the nuclear norm ball has the structure of a spectrahedron, and semidefinite programming duality allows for linear optimization over the operator norm ball. The spectrahedral descriptions of both unit balls in $\mathbb{R}^{n\times n}$ are special instances of Theorem 4.7, to be stated explicitly once more. The operator norm ball consists of all matrices $X$ whose largest singular value is at most 1. This is equivalent to

$$\begin{pmatrix} \mathrm{Id}_n & X \\ X^T & \mathrm{Id}_n \end{pmatrix} \succeq 0. \tag{4.4}$$

The nuclear norm ball consists of all matrices $X$ for which the sum of the singular values is at most 1. This is equivalent to saying that the sum of the largest $n$ eigenvalues of the symmetric $\binom{2n}{n} \times \binom{2n}{n}$ matrix $\mathcal{L}_n(S(X))$ is at most 1. Hence the nuclear norm ball in $\mathbb{R}^{n\times n}$ is the spectrahedron defined by the linear matrix inequality

$$\mathrm{Id}_{\binom{2n}{n}} - \mathcal{L}_n(S(X)) \,\succeq\, 0. \tag{4.5}$$

We are now finally prepared to return to the main aim of this section, which is the study of tautological orbitopes. The proof of Theorem 4.3 implies that the convex hull of $O(n)$ equals the set of $n\times n$-matrices whose largest singular value is at most 1. In other words:

**Corollary 4.9.** *The operator norm ball in* $\mathbb{R}^{n\times n}$ *is equal to the tautological orbitope of the orthogonal group* $O(n)$*. The coorbitope* $\mathrm{conv}(O(n))^\circ$ *is the nuclear norm ball. Both of these convex bodies are spectrahedra. They are characterized in (4.4) and (4.5) respectively.*

It would be worthwhile to explore implications of our geometric explorations of orbitopes for algorithmic applications in the sciences and engineering, such as those proposed in [31].

4.4. **The special orthogonal group.** We next discuss the faces of the tautological orbitope of the group $SO(n)$. The relevant convex polytope is now the convex hull $\mathsf{HC}_n$ of all vertices $v \in \{-1,+1\}^n$ with an even number of $(-1)$-entries. This polytope is known as the *demicube* or *halfcube*. It is a permutahedron for the Coxeter group of type $D_n$, i.e. signed permutations with an even number of sign changes. The facet hyperplanes of $\mathsf{HC}_n$ are derived by separating infeasible vertices of $[-1,+1]^n$ by hyperplanes through the $n$ neighboring vertices:

$$(4.6)\quad \mathsf{HC}_n \;=\; \Big\{x \in [-1,+1]^n \,:\, \sum_{i\notin J} x_i - \sum_{i\in J} x_i \leq n-2 \;\text{ for all } J \subseteq [n] \text{ of odd cardinality}\Big\}.$$

While $\mathsf{HC}_2$ is only a line segment, we have $\dim(\mathsf{HC}_n) = n$ for $n \geq 3$. For example, the halfcube $\mathsf{HC}_3$ is the tetrahedron with vertices $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$ and $(1,-1,-1)$. Note that its facet inequalities appear as the diagonal entries in the symmetric $4{\times}4$-matrix (4.1):

$$\mathsf{HC}_3 \;=\; \big\{\, x \in \mathbb{R}^3 \,:\, \min(1{+}x_1{+}x_2{+}x_3,\, 1{+}x_1{-}x_2{-}x_3,\, 1{-}x_1{+}x_2{-}x_3,\, 1{-}x_1{-}x_2{+}x_3) \geq 0 \,\big\}.$$

This observation is explained by results of Horn (cf. [22]) on the diagonals of special orthogonal matrices. These imply the following lemma about the tautological orbitope of $SO(n)$:

**Lemma 4.10.** *The projection of* $\mathrm{conv}(SO(n))$ *onto the diagonal equals the halfcube* $\mathsf{HC}_n$.

*Proof.* Just like in Lemma 4.2, it is clear that $\mathsf{HC}_n$ is a subset of $\mathsf{D}(\mathrm{conv}(SO(n))$. The converse is derived from the linear algebra fact that the trace of any matrix in $O(n)\backslash SO(n)$ is at most $n-2$. For $J \subseteq [n]$ let $R_J$ be the diagonal matrix with $(R_J)_{ii} = -1$ if $i \in J$ and $(R_J)_{ii} = 1$ if $i \notin J$. Let $g \in SO(n)$. Then $\mathrm{trace}(g \cdot R_J) \leq n-2$ for all $J$ of odd cardinality. This means that $\mathsf{D}(g)$ satisfies the linear inequalities in (4.6) and hence lies in $\mathsf{HC}_n$. □

There is a variant of singular value decomposition with respect to the restricted class of orientation preserving transformations. For every matrix $A \in \mathbb{R}^{n\times n}$ there exist rotations $U, V \in SO(n)$ such that $UAV$ is diagonal. The diagonal entries are called the *special singular values* and denoted by $\tilde\sigma(A) = (\tilde\sigma_1(A) \geq \cdots \geq \tilde\sigma_n(A))$. The main difference to the usual singular values is that $\tilde\sigma_n(A)$ may be negative; only the first $n-1$ entries of $\tilde\sigma(A)$ are non-negative. We need to make this distinction in order to understand the faces of $\mathrm{conv}(SO(n))$.

**Theorem 4.11.** *The tautological orbitope of* $SO(n)$ *has precisely two orbits of facets. These are the tautological orbitopes for* $SO(n-1)$ *and the free spectrahedra of dimension* $\binom{n}{2} - 1$.

*Proof.* Up to $D_n$-symmetry, the halfcube $\mathsf{HC}_n$ has only two distinct facets, namely an $(n-1)$-dimensional halfcube and an $(n-1)$-dimensional simplex. A typical halfcube facet $(\mathsf{HC}_n)^\ell$ arises by maximizing the linear function $\ell = x_1$ over $\mathsf{HC}_n$, and a typical simplex facet $(\mathsf{HC}_n)^{\ell'}$ arises by maximizing the linear function $\ell' = x_1{+}x_2{+}\cdots{+}x_{n-1}{-}x_n$. Pulling back $\ell$ along the diagonal projection $\mathsf{D}$, we see that the facet of $\mathrm{conv}(SO(n))$ corresponding to the halfcube facet is the convex hull of all rotations $g \in SO(n)$ that fix the first standard basis vector $e_1$. Pulling back $\ell'$ along $\mathsf{D}$, we see that the facet of $\mathrm{conv}(SO(n))$ corresponding to the simplex facet is the convex hull of $\big\{g \in SO(n) \,:\, \mathrm{Tr}(g \cdot R_{\{n\}}) = n-2\big\}$. This facet is isomorphic to the convex hull of all $g' \in O(n)\backslash SO(n)$ such that $\mathrm{Tr}(g') = n-2$. Since $g'$ is orientation reversing, one eigenvalue of $g'$ is $-1$, and $\mathrm{Tr}(g') = n-2$ forces all other eigenvalues to be equal to 1. Hence the facet in question is the symmetric Schur-Horn orbitope for the diagonal matrix $(1,\ldots,1,-1)$. Example 3.7 implies that this is a free spectrahedron. □

In Subsection 4.1 we exhibited a spectrahedral representation for the tautological orbitope $\mathrm{conv}(SO(3))$, and in that case, the two facet types of Theorem 4.11 collapse into one type. At present, we do not know how to generalize the representation (4.1) to $SO(n)$ for $n \geq 4$.

## 5. Carathéodory Orbitopes

Orbitopes for $SO(2)$ were first studied by Carathéodory [8]. The coorbitope cone (2.3) dual to such a *Carathéodory orbitope* consists of non-negative trigonometric polynomials. This leads to the Toeplitz spectrahedral representation of the universal Carathéodory orbitope in Theorem 5.2, which implies that the convex hulls of all trigonometric curves are projections of spectrahedra. The universal Carathéodory orbitope is also affinely isomorphic to the convex hull of the compact even moment curve, whose coorbitope cone consists of non-negative univariate polynomials. This leads to the representation by Hankel matrices in Theorem 5.6.

5.1. **Toeplitz representation.** The irreducible representations $\rho_a$ of $SO(2)$ are indexed by non-negative integers $a$. Here, $\rho_0$ is the trivial representation. When $a \in \mathbb{N}$ is positive, the representation $\rho_a$ of $SO(2)$ acts on $\mathbb{R}^2$, and it sends a rotation matrix to its $a$th power:

$$\rho_a \ : \ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \ \mapsto \ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^a = \begin{pmatrix} \cos(a\theta) & -\sin(a\theta) \\ \sin(a\theta) & \cos(a\theta) \end{pmatrix}.$$

For a vector $A = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$ we consider the direct sum of these representations

$$\rho_A \ := \ \rho_{a_1} \oplus \rho_{a_2} \oplus \cdots \oplus \rho_{a_d}.$$

The *Carathéodory orbitope* $\mathcal{C}_A$ is the convex hull of the orbit $SO(2) \cdot (1,0)^d$ under the action $\rho_A$ on the vector space $(\mathbb{R}^2)^d$. This orbit is the *trigonometric moment curve*

$$\Big\{ \big( \cos(a_1\theta), \sin(a_1\theta), \ldots, \cos(a_d\theta), \sin(a_d\theta) \big) \in \mathbb{R}^{2d} \ : \ \theta \in [0, 2\pi) \Big\}. \tag{5.1}$$

This curve is also identified with the matrix group $\rho_A(SO(2))$ lying in the space $(\mathbb{R}^{2\times 2})^d$ of block-diagonal $2d \times 2d$-matrices with $d$ blocks of size $2 \times 2$. Thus $\mathcal{C}_A$ is isomorphic to the convex hull of $\rho_A(SO(2))$, and can therefore also be thought of as a tautological orbitope.

We distinguish between isomorphisms of orbitopes that preserve the $SO(2)$-action, and the weaker notion of affine isomorphisms that preserve their structure as convex bodies.

**Lemma 5.1.** *Any orbitope of the circle group $SO(2)$ is isomorphic to a Carathéodory orbitope $\mathcal{C}_A$, where $A \in \mathbb{N}^d$ has distinct coordinates, and it is affinely isomorphic to a Carathéodory orbitope where the coordinates of $A$ are relatively prime integers.*

*Proof.* Let $\mathcal{O}$ be an orbitope for $SO(2)$. We may assume that its ambient $SO(2)$-module $V$ has no trivial components and is the linear span of $\mathcal{O}$. Then $V$ has the form $\rho_A$ for $A \in \mathbb{N}^d$ with distinct non-zero components, that is, $V$ is multiplicity free. This is because $\mathrm{End}_{SO(2)}(\rho_a) = \mathbb{C}$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$ and $SO(2)$ with the unit circle in $\mathbb{C}$. The orbitope $\mathcal{O}$ is generated by a vector $v = (v_1, \ldots, v_d) \in \mathbb{C}^d$ with non-zero coordinates. By complex rescaling, we may assume that $v = (1, \ldots, 1)$, showing that $\mathcal{O}$ is isomorphic to $\mathcal{C}_A$. Lastly, if the coordinates of $A = (a_1, \ldots, a_d)$ have greatest common divisor $a$, then $\rho_A$ is the composition $\rho_{A'} \circ \rho_a$, where $A' = (a_1/a, \ldots, a_d/a)$ and $\mathcal{O}$ is affinely isomorphic to an orbitope for the module $\rho_{A'}$. □

We henceforth assume that $0 < a_1 < \cdots < a_d$ where the $a_i$ are relatively prime. When $A = (1, 2, \ldots, d)$, Carathéodory [8] studied the facial structure of $\mathcal{C}_A$. An even-dimensional cyclic polytope is the convex hull of finitely many points on Carathéodory's curve (see [38]). Smilansky [34] studied the four-dimensional Carathéodory orbitopes ($d = 2$), and this was recently extended by Barvinok and Novik [5] who studied $\mathcal{C}_{(1,3,5,\ldots,2k-1)}$. The corresponding curve (5.1) is the *symmetric moment curve* which gives rise to a remarkable family of centrally symmetric polytopes with extremal face numbers. Many questions remain about the facial structure of the Barvinok-Novik orbitopes $\mathcal{C}_{(1,3,5,\ldots,2k-1)}$. See [5, §7.4] for details.

We now focus on the *universal Carathéodory orbitope* $\mathcal{C}_d := \mathcal{C}_{(1,2,\ldots,d)}$ in $\mathbb{R}^{2d}$. This convex body has the following spectrahedral representation in terms of Hermitian Toeplitz matrices.

**Theorem 5.2.** *The universal Carathéodory orbitope $\mathcal{C}_d$ is isomorphic to the spectrahedron consisting of positive semidefinite Hermitian Toeplitz matrices with ones along the diagonal:*

$$\begin{pmatrix} 1 & x_1 & \cdots & x_{d-1} & x_d \\ y_1 & 1 & \ddots & x_{d-2} & x_{d-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ y_{d-1} & y_{d-2} & \ddots & 1 & x_1 \\ y_d & y_{d-1} & \cdots & y_1 & 1 \end{pmatrix} \succeq 0 \qquad \textit{where} \qquad \begin{bmatrix} x_j = c_j + s_j \cdot i \\ y_j = c_j - s_j \cdot i \\ i = \sqrt{-1} \end{bmatrix} \tag{5.2}$$

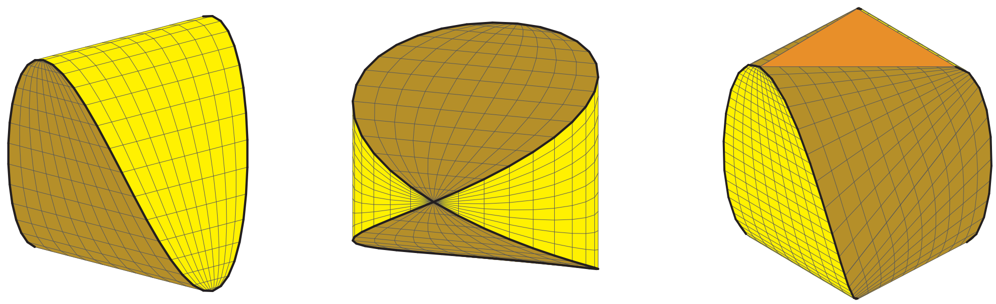
We note that the complex spectrahedron (5.2) can be translated into a spectrahedron over $\mathbb{R}$ as follows. Consider a Hermitian matrix $H = F + G \cdot i$ where $F$ is real symmetric and $G$ is real skew-symmetric. The Hermitian matrix $H$ is positive definite if and only if

$$\begin{pmatrix} F & -G \\ G & F \end{pmatrix} \succeq 0.$$

A *trigonometric curve* is any curve in $\mathbb{R}^n$ that is parametrized by polynomials in the trigonometric functions sine and cosine, or equivalently, any curve that is the image under a linear map of the universal trigonometric moment curve (5.1) where $A = (1, 2, \ldots, d)$.

**Corollary 5.3** (cf. Henrion [19])**.** *The convex hull of any trigonometric curve is a projected spectrahedron. In particular, all Carathéodory orbitopes are projected spectrahedra.*

Figure 3 shows the convex hulls of three trigonometric curves in $\mathbb{R}^3$. The left and middle convex bodies are each the intersection of two convex quadratic cylinders ($2x^2 = 1 + z$ and $2y^2 = 1 - z$ for the former; $x^2 + y^2 = 1$ and $2z^2 = 1 + x$ for the latter) and hence are spectrahedra. The rightmost convex body is visibly not a spectrahedron. Its exposed points are $(\cos(\theta), \sin(2\theta), \cos(3\theta))$ for $\theta \in (-\frac{\pi}{3}, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, \frac{4\pi}{3})$, and it has two algebraic families of one-dimensional facets. In addition, there are two two-dimensional facets, namely the equilateral triangles for $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\theta = \pi, \frac{\pi}{3}, -\frac{\pi}{3}$. The six edges of these two triangles are one-dimensional non-exposed faces, as there is no linear function which achieves its minimum on this body along these edges. Also, exactly one vertex of each triangle at $\theta = 0$ and $\theta = \pi$ is exposed. We conclude that this convex body is a projected spectrahedron but it is not a spectrahedron.



$(\cos(\theta), \sin(\theta), \cos(2\theta))$ $(\cos(2\theta), \sin(2\theta), \cos(\theta))$ $(\cos(\theta), \sin(2\theta), \cos(3\theta))$

FIGURE 3. Convex hulls of three trigonometric curves.

*Proof of Theorem 5.2.* The coorbitope cone dual to $\mathcal{C}_d$ consists of all affine-linear functions that are non-negative on $\mathcal{C}_d$. These correspond to non-negative trigonometric polynomials:

$$\widehat{\mathcal{C}}_d^\circ \;=\; \bigl\{(\delta, a_1, b_1, \ldots, a_d, b_d) \in \mathbb{R}^{2d+1} : \delta + \sum_{k=1}^{d} a_k \cos(k\theta) + b_k \sin(k\theta) \geq 0 \text{ for all } \theta\bigr\}.$$

We identify each point $(\delta, a_1, b_1, \ldots, a_d, b_d)$ in $\mathbb{R}^{2d+1}$ with the Laurent polynomial

$$R(z) \;=\; \sum_{k=-d}^{d} u_k z^k \in \mathbb{C}[z, z^{-1}] \tag{5.3}$$

with $u_0 = \delta$, $u_k = \frac{1}{2}(a_k - b_k i)$, $i = \sqrt{-1}$, and $u_{-k} = \overline{u_k}$. Then $\overline{R(z)} = R(\overline{z}^{-1})$ and $R \in \widehat{\mathcal{C}}_d^\circ$ if and only if $R$ is non-negative on the unit circle $\mathbb{S}^1$ of $\mathbb{C}$. Roots of $R$ occur in pairs $\alpha, \overline{\alpha}^{-1}$ and those on $\mathbb{S}^1$ have even multiplicity. Choosing one root from each pair gives the factorization

$$R(z) \;=\; \overline{H}(z^{-1}) \cdot H(z), \tag{5.4}$$

where $H \in \mathbb{C}[z]$ has degree $d$ and the coefficient vectors of $H$ and $\overline{H}$ are complex conjugate. This factorization is the classical Fejér-Riesz Theorem.

Utilizing the monomial map $\gamma_d : \mathbb{C} \to \mathbb{C}^{d+1}$ with $\gamma_d(z) = (1, z, z^2, \ldots, z^d)^T$, this is equivalent to the following: A trigonometric polynomial $R(z)$ is non-negative on the unit circle if and only if there is a non-zero vector $h \in \mathbb{C}^{d+1}$ such that

$$R(z) \;=\; \gamma_d(z^{-1})^T \cdot \overline{h}h^T \cdot \gamma_d(z). \tag{5.5}$$

A point $(c_1, s_1, \ldots, c_d, s_d) \in \mathbb{R}^{2d}$ belongs to the Carathéodory orbitope $\mathcal{C}_d$ if and only if

$$\delta + \sum_{k=1}^{d} a_k c_k + b_k s_k \;\geq\; 0 \quad \text{ for all } (\delta, a_1, b_1, \ldots, a_d, b_d) \in \widehat{\mathcal{C}}_d^\circ. \tag{5.6}$$

The sum on the left equals the Hermitian inner product in $\mathbb{C}^{2d+1}$ of the coefficient vector $u$ of the polynomial $R(z)$ and the vector $\zeta = (x, 1, y)$ with $x_k, y_k$ as in (5.2). The formula (5.5) expresses $u$ as the image of the Hermitian matrix $\overline{h}h^T$ under some linear projection $\pi$. If $\pi^*$

denotes the linear map dual to $\pi$ then $X = \pi^*(\zeta)$ is precisely the Hermitian Toeplitz matrix in (5.2). We conclude that the sum in (5.6) equals

$$\langle \zeta, \pi(\overline{h}h^T)\rangle \;=\; \langle \pi^*(\zeta), \overline{h}h^T\rangle \;=\; \mathrm{Tr}(X \cdot \overline{h}h^T) \;=\; h^T \cdot X \cdot \overline{h}.$$

Thus the point $(c_1, s_1, \ldots, c_d, s_d)$ represented by a Hermitian Toeplitz matrix $X$ lies in $\mathcal{C}_d$ if and only if $h^T \cdot X \cdot \overline{h} \geq 0$ for all $h \in \mathbb{C}^{d+1}$ if and only if $X$ is positive semidefinite. $\square$

The proof of Theorem 5.2 elucidates the known results about the facial structure of $\mathcal{C}_d$.

**Corollary 5.4.** *The universal Carathéodory orbitope $\mathcal{C}_d$ is a neighborly simplicial convex body. Its faces are in inclusion-preserving correspondence with sets of at most $d$ points on the circle.*

*Proof.* A Laurent polynomial $R$ as in (5.3) lies in the boundary of the coorbitope cone $\widehat{\mathcal{C}}_d^\circ$ if and only if it is non-negative on the unit circle $\mathbb{S}^1$ but not strictly positive. It supports the face of $\mathcal{C}_d$ spanned by the points of the trigonometric moment curve corresponding to its zeros in $\mathbb{S}^1$. Each zero has multiplicity at least 2, so there are at most $d$ such points, and conversely any subset of $\leq d$ points supports a face. Since any fewer than $2d+2$ points on the curve are affinely independent and since all faces are exposed, these faces are simplices. $\square$

Corollary 5.4 implies that the Carathéodory number $\mathfrak{c}(\mathcal{C}_d)$ equals $d+1$, as no point in the interior of $\mathcal{C}_d$ lies in the convex hull of $d$ points of the orbit, but $\mathfrak{c}(\mathcal{C}_d)$ is at most one more than the maximal Carathéodory number of a facet, by Lemma 3.2 of [23].

Carathéodory orbitopes are generally not spectrahedra because they can possess non-exposed faces. Smilansky [34] showed that if we write $\rho(\theta) \in \mathbb{R}^4$ for a point on the trigonometric moment curve with weights 1 and 3 then the faces of $\mathcal{C}_{(1,3)}$ are exactly the points $\rho(\theta)$ of the orbit, the line segments $\mathrm{conv}\{\rho(\theta), \rho(\theta+\alpha)\}$, where $0 < \alpha < \frac{2\pi}{3}$, and the triangles

$$\mathrm{conv}\{\rho(\theta),\ \rho(\theta + 2\pi/3),\ \rho(\theta+4\pi/3)\}.$$

In particular, each line segment $\mathrm{conv}\{\rho(\theta), \rho(\theta + \frac{2\pi}{3}\}$ is a non-exposed edge of $\mathcal{C}_{(1,3)}$. We conclude that the Barvinok-Novik orbitope $\mathcal{C}_{(1,3)}$ is not a spectrahedron.

The Toeplitz representation (5.2) of the universal Carathéodory orbitope $\mathcal{C}_d$ reveals complete algebraic information. For example, the algebraic boundary $\partial_a \mathcal{C}_d$ is the irreducible hypersurface of degree $d+1$ defined by the determinant of that $(d{+}1) \times (d{+}1)$-matrix. The curve (5.1) itself is the set of all positive definite Hermitian Toeplitz matrices of rank one. The $2 \times 2$-minors of the matrix (5.2) generate the prime ideal $J_{(1,\ldots,d)}$ of this rational curve.

The union of the $(j-1)$-dimensional faces of $\mathcal{C}_d$ is the set of positive definite Hermitian Toeplitz matrices of rank $j$, as a point lies on a $(j-1)$-dimensional face if and only if it is the convex combination of $j$ points of the curve. The Zariski closure of this stratum is the set of all rank $j$ Hermitian Toeplitz matrices which is defined by the vanishing of the $(j{+}1) \times (j{+}1)$ minors of that matrix. This is also the $j$th secant variety of the Carathéodory curve. Lastly, this rank stratification is a Whitney stratification of the algebraic boundary.

The derivation of the algebraic description of $\mathcal{C}_A$ for arbitrary $A = (a_1, a_2, \ldots, a_d)$ requires the process of elimination. For instance, the ideal $J_A$ of the trigonometric moment curve

(5.1) can be computed from the ideal of 2×2-minors for $J_{(1,2,\ldots,a_d)}$ by eliminating all unknowns $x_j, y_j$ with $j \notin \{a_1, \ldots, a_d\}$. The equation of the algebraic boundary $\partial_a \mathcal{C}_A$ is obtained by the same elimination applied to a certain ideal of larger minors of the Toeplitz matrix (5.2).

We refer to recent work of Vinzant [36] for a detailed study of the edges of Barvinok-Novik orbitopes. An analysis of the algebraic boundary the orbitope $\mathcal{C}_{(1,3)}$ can be found in [30, §2.4].

5.2. **Hankel representation.** The cone over the degree $d$ moment curve is the image of $\mathbb{R}^2$ in its $d$th symmetric power $\mathrm{Sym}_d\mathbb{R}^2 \simeq \mathbb{R}^{d+1}$ under the map

$$\nu_d \;:\; (x,y) \;\longmapsto\; \left(x^d,\, x^{d-1}y,\, x^{d-2}y^2,\, \ldots,\, y^d\right).$$

This map is naturally $SO(2)$-equivariant. We define the *compact moment curve* to be the image $\nu_d(\mathbb{S}^1)$ of the unit circle under the map $\nu_d$. This restricted map equals

$$\mathbb{S}^1 \ni \theta \;\longmapsto\; (\cos^d(\theta), \cos^{d-1}(\theta)\sin(\theta), \ldots, \sin^d(\theta)).$$

The convex hull of the curve $\nu_d(\mathbb{S}^1)$ is an orbitope. By Lemma 5.1, it is isomorphic to some Carathéodory orbitope $\mathcal{C}_A$. The following lemma makes this identification explicit.

**Lemma 5.5.** *If $d \in \mathbb{N}$ is odd, then $\mathrm{conv}(\nu_d(\mathbb{S}^1))$ is isomorphic to the Barvinok-Novik orbitope $\mathcal{C}_{(1,3,\ldots,d)}$. If $d \in \mathbb{N}$ is even, then $\mathrm{conv}(\nu_d(\mathbb{S}^1))$ is isomorphic to $\mathcal{C}_{(0,2,4,\ldots,d)}$, which is affinely isomorphic to the universal Carathéodory orbitope $\mathcal{C}_{d/2}$.*

*Proof.* Complexifying the $SO(2)$-module $\rho_A$ where $A = (a_1, \ldots, a_d)$ gives the $\mathbb{C}^\times$-module with symmetric weights $\pm a_1, \ldots, \pm a_d$. Thus the underlying real $SO(2)$-module of this $\mathbb{C}^\times$-module is $\rho_{(|a_1|,\ldots,|a_d|)}$. The lemma follows because the complexified representation $\mathrm{Sym}_d\mathbb{C}^2$ of $\mathrm{Sym}_d\mathbb{R}^2$ has weights $d, d{-}2, d{-}4, \ldots, -d$, and this representation is spanned by the pure powers of linear forms, so every weight appears in the linear span of the orbit. □

Suppose now that $d = 2n$ is even. We describe the moment curve and the Carathéodory orbitope in coordinates $(\lambda_0, \lambda_1, \ldots, \lambda_{2n})$ for $\mathbb{R}^{2n+1}$. Fix the $(n{+}1)\times(n{+}1)$-Hankel matrix

$$K(\lambda) \quad = \quad \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n+1} \\ \lambda_2 & \lambda_3 & \lambda_4 & \cdots & \lambda_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_n & \lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{2n} \end{pmatrix}. \tag{5.7}$$

**Theorem 5.6.** *The even moment curve consists of all vectors $\lambda \in \mathbb{R}^{2n+1}$ such that the Hankel matrix $K(\lambda)$ has rank one, is positive semidefinite, and satisfies the linear equation*

$$\sum_{j=0}^{n} \binom{n}{j} \lambda_{2j} \;=\; 1. \tag{5.8}$$

*Its convex hull consists of all $\lambda$ such that $K(\lambda)$ is positive semidefinite and satisfies* (5.8).

Thus the Carathéodory orbitope $\mathcal{C}_n$ has a second *Hankel representation* as a spectrahedron. The proof of this theorem follows from the well-known fact that every non-negative polynomial in one variable is a sum of squares of polynomials. It uses the duality in [32, §3].

*Proof.* Observe that the points of the even compact moment curve satisfy (5.8), which comes from the polynomial identity $(x^2+y^2)^n = 1$. As $\mathrm{Sym}_{2n}\mathbb{R}^2$ has just one copy of the trivial representation, this is the only affine equation that holds on $\nu_{2n}(\mathbb{S}^1)$. The dual to $\mathrm{Sym}_{2n}\mathbb{R}^2$ is the space $\mathbb{R}[x,y]_{2n}$ of real homogeneous polynomials of degree $2n$ in $x$ and $y$. The coefficients of such a polynomial gives coordinates for $\mathbb{R}[x,y]_{2n}$.

The coorbitope cone $\widehat{\nu_{2n}(\mathbb{S}^1)^\circ}$ dual to the orbitope $\mathrm{conv}(\nu_{2n}(\mathbb{S}^1))$ is the cone of homogeneous polynomials of degree $2n$ that are non-negative on $\mathbb{R}^2$. Thus a point $(\lambda_0,\ldots,\lambda_{2n}) \in \mathrm{Sym}_{2n}\mathbb{R}^2$ lies in the orbitope $\mathrm{conv}(\nu_{2n}(\mathbb{S}^1))$ if and only if it satisfies (5.8) and

$$\sum_{i=0}^{2n} f_i\lambda_i \;\geq\; 0 \tag{5.9}$$

for every non-negative polynomial $f(x,y) = \sum_i f_i x^i y^{2n-i}$.

Since non-negative homogeneous polynomials in $x$ and $y$ are sums of squares (cf. [32]), we only need these inequalities to hold when $f(x,y) = g(x,y)^2$ is a square. Writing $g = (g_0,\ldots,g_n)$ for the coefficient vector of the polynomial $g(x,y)$, the sum (5.9) becomes

$$\sum_{i=0}^{2n} \lambda_i \sum_{j+k=i} g_j g_k \;=\; g^T \cdot K(\lambda) \cdot g\,,$$

where $K(\lambda)$ is the Hankel matrix (5.7). This proves the theorem. □

## 6. Veronese Orbitopes

The Hankel representation of the universal Carathéodory orbitope arose by considering the image of the circle $\mathbb{S}^1 \subset \mathbb{R}^2$ in $\mathrm{Sym}_{2n}\mathbb{R}^2$ and its relation to non-negative binary forms. Generalizing from $\mathbb{R}^2$ to $\mathbb{R}^d$ gives the Veronese orbitopes whose coorbitope cones (2.3) consist of non-negative $d$-ary forms. When $d \geq 3$, non-negative forms are not necessarily sums of squares, except for quadratic forms and the exceptional case of ternary quartics.

The set of decomposable symmetric tensors is the image of the Veronese map

$$\nu_m \,:\; \mathbb{R}^d \;\longrightarrow\; \mathrm{Sym}_m\mathbb{R}^d \;\simeq\; \mathbb{R}^{\binom{d+m-1}{m-1}}\,.$$

The $SO(d)$-orbits through any two non-zero decomposable tensors are scalar multiples of each other and are thus isomorphic. We define the *Veronese orbitope* $\mathcal{V}_{d,m}$ to be the convex hull of the orbit through the specific decomposable tensor $\nu_m(1,0,\ldots,0)$. That orbit is also the image $\nu_m(\mathbb{S}^{d-1})$ of the unit $(d-1)$-sphere under the $m$-th Veronese embedding of $\mathbb{R}^d$.

Suppose that $m = 2n$ is even. Then the orbit $\nu_m(\mathbb{S}^{d-1})$ can be identified with $\mathbb{RP}^{d-1}$ since $\nu_{2n}$ is two-to-one with $\nu_{2n}(v) = \nu_{2n}(-v)$. The dual vector space to $\mathrm{Sym}_{2n}\mathbb{R}^d$ is the space of homogeneous forms of degree $2n$ on $\mathbb{R}^d$. The only invariant forms are those proportional to the form $\langle v, v\rangle^n$, so both $\mathrm{Sym}_{2n}\mathbb{R}^d$ and its dual space contain one copy of the trivial representation, and $\nu_{2n}(\mathbb{S}^{d-1})$ lies in the hyperplane of $\mathrm{Sym}_{2n}\mathbb{R}^d$ defined by $\langle v, v\rangle^n = 1$.

The dual cone to the Veronese orbitope $\mathcal{V}_{d,2n} = \mathrm{conv}(\nu_{2n}(\mathbb{S}^{d-1}))$ is the cone of non-negative forms of degree $2n$ in $\mathrm{Sym}_{2n}(\mathbb{R}^d)^*$. See also [4, Example (1.2)]. We write $\widehat{\mathcal{V}}^\circ_{d,2n}$ for the Veronese coorbitope cone consisting of non-negative forms. The cone $\widehat{\mathcal{V}}^\circ_{d,2n}$ of non-negative forms contains the cone $\mathcal{K}_{d,2n}$ of sums of squares, but when $d \geq 3$, $2n \geq 4$, and $(d, 2n) \neq (3, 4)$, Hilbert [20] showed that the inclusion is strict. We refer to [7] for a recent study which compares the dimension of the faces of these cones.

The cone $\mathcal{K}_{d,2n}$ is naturally a projection of the positive semidefinite cone. Hence its dual cone $\mathcal{C}_{d,2n} = \mathcal{K}^\circ_{d,2n}$ is a spectrahedron: it can be realized as the intersection of the positive semidefinite cone with a certain linear space of generalized Hankel matrices, discussed in detail in Reznick's book [32]. This spectrahedron $\mathcal{C}_{d,2n}$ is strictly larger than the Veronese orbitope $\mathcal{V}_{d,2n}$ when $d \geq 3$, $2n \geq 4$, and $(d, 2n) \neq (3, 4)$. In fact, $\mathcal{V}_{d,2n}$ is precisely the convex hull of the subset of extreme points in $\mathcal{C}_{d,2n}$ that have rank one.

We present a detailed case study of the exceptional case of ternary quartics, when $(d, 2n) = (3, 4)$. The Veronese orbitope $\mathcal{V}_{3,4} = \mathrm{conv}(\nu_4(\mathbb{R}^3))$ is a 14-dimensional convex body. Let $\widehat{\mathcal{V}}_{3,4}$ be the 15-dimensional cone over the Veronese orbitope $\mathcal{V}_{3,4}$. As all non-negative ternary quartics are sums of squares, we have the following identities of cones:

$$\widehat{\mathcal{V}}_{3,4} = \mathcal{C}_{3,4} \quad \text{and} \quad \widehat{\mathcal{V}}^\circ_{3,4} = \mathcal{K}_{3,4}.$$

We next present Reznick's spectrahedral representation of $\mathcal{V}_{3,4}$. For this, we identify $\mathrm{Sym}_4\mathbb{R}^3$ with its dual, and we introduce coordinates $\lambda = (\lambda_\alpha)$ where the indices $\alpha$ are the exponents of monomials in variables $x, y, z$ of degree 4. The ternary quartic corresponding to $\lambda$ is

$$q_\lambda = \sum_\alpha \binom{4}{\alpha} \lambda_\alpha x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}, \tag{6.1}$$

where $\binom{4}{\alpha} = \frac{4!}{\alpha_1!\alpha_2!\alpha_3!}$ is the multinomial coefficient. The inner product $\langle q_\lambda, q_\mu\rangle = \sum_\alpha \binom{4}{\alpha}\lambda_\alpha\mu_\alpha$ is $SO(3)$-invariant. Given a ternary quartic $q_\lambda$ in the notation (6.1), we associate to it the following symmetric $6 \times 6$-matrix with Hankel structure as in [32, eqn. (5.25)]:

$$K_\lambda = \begin{pmatrix} \lambda_{400} & \lambda_{220} & \lambda_{202} & \lambda_{310} & \lambda_{301} & \lambda_{211} \\ \lambda_{220} & \lambda_{040} & \lambda_{022} & \lambda_{130} & \lambda_{121} & \lambda_{031} \\ \lambda_{202} & \lambda_{022} & \lambda_{004} & \lambda_{112} & \lambda_{103} & \lambda_{013} \\ \lambda_{310} & \lambda_{130} & \lambda_{112} & \lambda_{220} & \lambda_{211} & \lambda_{121} \\ \lambda_{301} & \lambda_{121} & \lambda_{103} & \lambda_{211} & \lambda_{202} & \lambda_{112} \\ \lambda_{211} & \lambda_{031} & \lambda_{013} & \lambda_{121} & \lambda_{112} & \lambda_{022} \end{pmatrix}. \tag{6.2}$$

**Theorem 6.1.** *The Veronese orbitope $\mathcal{V}_{3,4}$ is a spectrahedron. It consists of all positive semidefinite Hankel matrices $K_\lambda$ as in (6.2) that satisfy the equation*

$$\lambda_{400} + \lambda_{040} + \lambda_{004} + 2\lambda_{220} + 2\lambda_{202} + 2\lambda_{022} = 1. \tag{6.3}$$

*Proof.* It is shown in [32, Ch. 5] that the quartic $q_\lambda$ is non-negative if and only if the Hankel matrix $K_\lambda$ is positive semidefinite. Furthermore, the equation (6.3) is the affine equation $(x^2+y^2+z^2)^2 = 1$ which defines the hyperplane containing the orbit $\nu_4(\mathbb{S}^2)$. □

By contrast, the Veronese coorbitope is not a spectrahedron.

**Theorem 6.2.** *The convex cone of non-negative ternary quartics $\mathcal{K}_{3,4} = \widehat{\mathcal{V}}^\circ_{3,4}$ is not a spectrahedron. Its facets have dimension twelve and the intersection of any two facets is an exposed face of dimension nine. It also has maximal non-exposed faces of dimension nine.*

*Proof.* We thank Greg Blekherman who explained this to us. The cone $\mathcal{K}_{3,4}$ is full-dimensional in the 15-dimensional space $\mathrm{Sym}_{2n}(\mathbb{R}^d)^*$. Its facets come from its defining linear inequalities

$$\mathcal{K}_{3,4} \;=\; \left\{q \in \mathrm{Sym}_{2n}(\mathbb{R}^d)^* \mid q(p) \geq 0 \quad \forall p \in \mathbb{RP}^2\right\}.$$

For $p \in \mathbb{RP}^2$, let $F^p$ be the facet exposed by the inequality $q(p) \geq 0$, which consists of those non-negative ternary quartics $q$ that vanish at $p$. Since the boundary of $\mathcal{K}_{3,4}$ is 14-dimensional and we have a two-dimensional family of isomorphic facets, we see that each facet $F^p$ is 12-dimensional. A non-negative form that vanishes at $p \in \mathbb{RP}^2$ must also have its two partial derivatives (in local coordinates at $p$) vanish, which gives three linear conditions on the facet $F^p$. Concretely, if we take $p = [0:0:1]$ with local coordinates $x, y$, then the constant and linear terms of the inhomogeneous quartic $q(x,y)$ must vanish. Consequently,

$$q(x,y) \;=\; H(x,y) + C(x,y) + Q(x,y)\,, \tag{6.4}$$

where $H$, $C$, and $Q$ are, respectively, the terms of degrees 2, 3, and 4 in $q$. These are binary forms. Their $3+4+5 = 12$ coefficients parametrize the linear span of the facet $F^p$, showing again that $F^p$ is 12-dimensional. The quadratic form $H(x,y)$ is the Hessian of $q(x,y)$ at $p = [0:0:1]$ in these coordinates.

A form $q$ lies in the relative interior of the facet $F^p$ if and only if, given any $q'$ which vanishes at $p$ along with its partial derivatives, so that it lies in the linear span of $F^p$, there is an $\epsilon > 0$ such that $q + \epsilon q'$ also lies in $F^p$. These conditions are equivalent to

(1) $q$ has no other zeros in $\mathbb{RP}^2$, and
(2) the Hessian of $q$ at $p$ is a positive definite quadratic form.

A form $q \in F^p$ lies in the boundary of $F^p$ when one of these conditions fails, that is, either

(1) $q$ vanishes at a second point $p' \in \mathbb{RP}^2 \setminus \{p\}$, or
(2) the Hessian of $q$ has a double root at some point $r \in \mathbb{RP}^2$.

Faces of type (1) have the form $F^p \cap F^{p'}$. These are nine-dimensional and occur in a four-dimensional family parametrized by pairs of distinct points in $\mathbb{RP}^2$. The union of all such faces is a semialgebraic subset of dimension $9+4 = 13$ in the boundary of $\mathcal{K}_{3,4}$.

Faces of type (2) also have dimension nine. The condition that the Hessian has a double root at a point $r \in \mathbb{RP}^1$ gives two linear conditions on the coefficients of $q$. There is an additional condition that the cubic part $C$ of $q$ in (6.4) also vanishes at $r$, for otherwise $q$ takes negative values along the line through $p$ corresponding to $r$. A face of type (2) is the limit of faces $F^p \cap F^{p'}$ of type (1) as $p'$ approaches $p$ along the line corresponding to $r$. These faces form a three-dimensional family on which $SO(3)$ acts faithfully and transitively.

Let us now examine the exposed faces of $\mathcal{K}_{3,4}$. Let $\ell \in \widehat{\mathcal{V}}_{3,4}$ be a symmetric tensor in the dual cone to $\mathcal{K}_{3,4}$. Then $\ell$ is the sum of decomposable symmetric tensors,

$$\ell \;=\; \nu_4(p_1) + \cdots + \nu_4(p_s)\,,$$

and so it supports the face $F^{p_1} \cap \cdots \cap F^{p_s}$. Thus the exposed faces of $\mathcal{K}_{3,4}$ are intersections of facets, and they consist of non-negative ternary quartics that vanish at a given set of points.

Since the faces of type (2) are not of this form, they are not exposed. □

Our next agenda item is a discussion of the algebraic boundaries of the convex bodies and cones discussed above. The algebraic boundary of the cone $\widehat{\mathcal{V}}_{3,4}$ is characterized by the vanishing of the determinant of the Hankel matrix $K_\lambda$ in (6.2). From this we conclude:

**Corollary 6.3.** *The algebraic boundary of the Veronese orbitope* $\mathcal{V}_{3,4}$ *is the variety of dimension* 13 *and degree six which is defined by the linear equation* (6.3) *and the Hankel determinant* $\det(K_\lambda) = 0$. *The extreme points of* $\mathcal{V}_{3,4}$ *are precisely the Hankel matrices* $K_\lambda$ *of rank* 1.

We observed in the proof of Theorem 6.2 that the boundary of $\mathcal{K}_{3,4}$ consists of non-negative quartics $q$ that vanish at some point $p$ of $\mathbb{RP}^2$, and that the partial derivatives of $q$ necessarily also vanish at $p$. That is, the plane quartic curve defined by $q = 0$ is singular at $p$. Thus the algebraic boundary of $\mathcal{K}_{3,4}$ consists of singular ternary quartics. Working in $\mathbb{P}(\mathrm{Sym}_4\mathbb{C}^3) \simeq \mathbb{P}^{14}$, and its dual space of ternary quartics, this algebraic boundary is seen to be the dual variety to the Veronese surface which consists of rank 1 Hankel matrices $K_\lambda$.

**Corollary 6.4.** *The algebraic boundary of the coorbitope cone* $\mathcal{K}_{3,4}$ *is an irreducible hypersurface of degree* 27. *Its defining polynomial is the discriminant* $\Delta_q$ *of the ternary quartic*

$$\begin{aligned} q(x,y,z) \;=\;& c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + c_{220}x^2y^2 + c_{211}x^2yz + c_{202}x^2z^2 + c_{130}xy^3 \\ &+ c_{121}xy^2z + c_{112}xyz^2 + c_{103}xz^3 + c_{040}y^4 + c_{031}y^3z + c_{022}y^2z^2 + c_{013}yz^3 + c_{004}z^4\,. \end{aligned}$$

The discriminant $\Delta_q$ is a homogeneous polynomial of degree 27 in the 15 indeterminates $c_{ijk}$. In what follows we shall present an explicit expression for $\Delta_q$. That expression will be derived from a beautiful classical formula due to Sylvester which can be found in Section 3.4.D, starting on page 118, of the book by Gel′fand, Kapranov, and Zelevinsky [14].

According to [14, Prop. 1.7, page 434], the discriminant $\Delta_q$ is proportional to the resultant $R_3(q_x, q_y, q_x)$ of the three partial derivatives of the quartic $q$. Here $R_3$ denotes the resultant of three ternary cubics, and the precise relation is $\Delta_q \,=\, 4^{-7}\cdot R_3(q_x,q_y,q_x)$.

We write $(\mathbb{R}^3)^*$ for the space of linear forms on $\mathbb{R}^3$, and we introduce the linear map

$$T\;:\;(\mathbb{R}^3)^*\oplus(\mathbb{R}^3)^*\oplus(\mathbb{R}^3)^* \;\longrightarrow\; \mathrm{Sym}_4(\mathbb{R}^3)^*\,,\;\;(f,g,h)\mapsto fq_x+gq_y+hq_z. \qquad (6.5)$$

Next, for an exponent vector $\alpha = (\alpha_2, \alpha_2, \alpha_3)$ of degree $\alpha_1+\alpha_2+\alpha_3 = 2$ and any variable $t \in \{x,y,z\}$, we choose a decomposition of the cubic partial derivative

$$q_t \;=\; x^{\alpha_1+1}P^{(t)}_\alpha \;+\; y^{\alpha_2+1}Q^{(t)}_\alpha \;+\; z^{\alpha_3+1}R^{(t)}_\alpha\,, \qquad (6.6)$$

where $P^{(t)}_\alpha$, $Q^{(t)}_\alpha$, and $R^{(t)}_\alpha$ are forms of degree $2-\alpha_1$, $2-\alpha_2$, and $2-\alpha_3$, respectively. Then

$$D_\alpha \;=\; \det\begin{pmatrix} P^{(x)}_\alpha & Q^{(x)}_\alpha & R^{(x)}_\alpha \\ P^{(y)}_\alpha & Q^{(y)}_\alpha & R^{(y)}_\alpha \\ P^{(z)}_\alpha & Q^{(z)}_\alpha & R^{(z)}_\alpha \end{pmatrix}$$

is a quartic polynomial. Finally, we define a linear map $D\colon \mathrm{Sym}_2\mathbb{R}^3 \to \mathrm{Sym}_4(\mathbb{R}^3)^*$ by sending $\delta_\alpha \mapsto D_\alpha$, where $\{\delta_\alpha\}$ is the basis dual to the monomial basis of $\mathrm{Sym}_2(\mathbb{R}^3)^*$.

**Proposition 6.5.** (Sylvester [14, §3.4.D]) *The discriminant $\Delta_q$ is proportional to the resultant of the ternary cubics $q_x, q_y, q_z$ which is equal to the determinant of the linear map*

$$T \oplus D \;:\; (\mathbb{R}^3)^* \oplus (\mathbb{R}^3)^* \oplus (\mathbb{R}^3)^* \oplus \mathrm{Sym}_2(\mathbb{R}^3)^* \;\longrightarrow\; \mathrm{Sym}_4(\mathbb{R}^3)^* .$$

*This is an irreducible homogeneous polynomial of degree 27 in the 15 coefficients $c_{ijk}$.*

We write this map explicitly as a $15 \times 15$ matrix $\mathcal{D}(q)$ whose rows are indexed by the 15 monomials $x^iy^jz^k$ of degree $i+j+k=4$ and whose columns are indexed by 15 auxiliary quartics, and whose entry in a given row and column is the coefficient of that monomial in that auxiliary quartic. Nine of the quartics come from the map $T$ in (6.5). They are

$$xq_x\,,\; yq_x\,,\; zq_x\,,\; xq_y\,,\; yq_y\,,\; zq_y\,,\; xq_z\,,\; yq_z\,,\; zq_z\,.$$

The other six are the polynomials $D_{002}$, $D_{020}$, $D_{200}$, $D_{110}$, $D_{101}$, and $D_{011}$ from $D$. We only describe $D_{002}$ and $D_{110}$ as the others may be recovered from these by symmetry.

For $D_{002}$, note that each partial derivative of $q$ has six terms divisible by $x$, three divisible by $y$ and not $x$, and a unique term involving $z^3$. This leads to a decomposition (6.6), and $D_{002}$ is the determinant of the $3\times 3$ matrix:

$$\begin{bmatrix} 4c_{400}x^2+3c_{310}xy+3c_{301}xz+2c_{220}y^2+2c_{211}yz+2c_{202}z^2 & c_{130}y^2+c_{121}yz+c_{112}z^2 & c_{103} \\ c_{310}x^2+2c_{220}xy+c_{211}xz+3c_{130}y^2+2c_{121}yz+c_{112}z^2 & 4c_{040}y^2+3c_{031}yz+2c_{022}z^2 & c_{013} \\ c_{301}x^2+c_{211}xy+2c_{202}xz+c_{121}y^2+2c_{112}yz+3c_{103}z^2 & c_{031}y^2+2c_{022}yz+3c_{013}z^2 & 4c_{004} \end{bmatrix}$$

By a similar reasoning, we find that $D_{110}$ is the determinant of the $3\times 3$ matrix:

$$\begin{bmatrix} 4c_{400}x+3c_{310}y+3c_{301}z & 2c_{220}x+c_{130}y+c_{121}z & 2c_{211}xy+2c_{202}xz+c_{112}yz+c_{103}z^2 \\ c_{310}x+2c_{220}y+c_{211}z & 3c_{130}x+4c_{040}y+3c_{031}z & 2c_{121}xy+c_{112}xz+2c_{022}yz+c_{013}z^2 \\ c_{301}x+c_{211}y+2c_{202}z & c_{121}x+c_{031}y+2c_{022}z & 2c_{112}xy+3c_{103}xz+3c_{013}yz+4c_{004}z^2 \end{bmatrix}$$

This concludes our discussion of the algebraic boundary of the coorbitope cone $\mathcal{K}_{3,4}$.

We close with the remark that the notations $\mathcal{K}_{\bullet,\bullet}$ and $\mathcal{C}_{\bullet,\bullet}$ are consistent with those used in the paper [35] where these cones consist of concentration matrices and sufficient statistics of a certain Gaussian model.

## 7. Grassmann Orbitopes

The *Grassmann orbitope* $\mathcal{G}_{d,n}$ is the convex hull of the Grassmann variety of oriented $d$-dimensional linear subspaces of $\mathbb{R}^n$ in its Plücker embedding in the unit sphere in $\wedge_d \mathbb{R}^n$. Equivalently, this is the highest weight orbitope for the group $SO(n)$ acting on $\wedge_d \mathbb{R}^n$:

$$\mathcal{G}_{d,n} \;=\; \mathrm{conv}(SO(n) \cdot e_{12\cdots d}) \qquad \text{where } e_{12\cdots d} \;=\; e_1 \wedge e_2 \wedge \cdots \wedge e_d \;\in\; \wedge_d \mathbb{R}^n.$$

Faces of the Grassmann orbitope are of considerable interest in differential geometry since, according to the Fundamental Theorem of Calibrations, they correspond to area-minimizing $d$-dimensional submanifolds of $\mathbb{R}^n$. References to this subject include the seminal article on calibrated geometries by Harvey and Lawson [16] and the beautiful expositions by Morgan [17, 26]. In this section we review basic known facts about $\mathcal{G}_{d,n}$ and we initiate its study from the perspectives of combinatorics, semidefinite programming, and algebraic geometry.

Vectors in $\wedge_d \mathbb{R}^n$ are written in terms of Plücker coordinates relative to the standard basis:

$$p \;=\; \sum_{1 \le i_1 < \cdots < i_d \le n} p_{i_1 i_2 \cdots i_d} \, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}.$$

The Plücker vector $p$ lies in the *oriented Grassmann variety* if and only if it is decomposable, i.e. $p = u_1 \wedge u_2 \wedge \cdots \wedge u_d$ for some pairwise orthogonal subset $\{u_1, u_2, \ldots, u_d\}$ of $\mathbb{S}^{n-1}$. This happens if and only if $p$ lies in the unit sphere in $\wedge_d \mathbb{R}^n$ and satisfies all quadratic Plücker relations, the relations among the $d{\times}d$-minors of a $d{\times}n$-matrix. These relations generate the prime ideal called the *Plücker ideal* $I_{d,n}$. Thus the oriented Grassmann variety is the algebraic subvariety of $\wedge_d \mathbb{R}^n$ defined by the ideal

$$I_{d,n} \;+\; \Big\langle\, 1 - \sum_{1 \le i_1 < \cdots < i_d \le n} p_{i_1 i_2 \cdots i_d}^2 \,\Big\rangle. \tag{7.1}$$

The convex hull of that real algebraic variety is the $\binom{n}{d}$-dimensional Grassmann orbitope $\mathcal{G}_{d,n}$.

**Example 7.1** ($d$=2, $n$=4)**.** The Grassmann orbitope $\mathcal{G}_{2,4}$ is the convex hull of the variety of

$$\big\langle\, p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}\,,\; p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{23}^2 + p_{24}^2 + p_{34}^2 - 1 \,\big\rangle. \tag{7.2}$$

As suggested by [26, Proposition 3.2], we perform the orthogonal change of coordinates

$$u = \tfrac{1}{\sqrt{2}}(p_{12} + p_{34}), \quad v = \tfrac{1}{\sqrt{2}}(p_{13} - p_{24}), \quad w = \tfrac{1}{\sqrt{2}}(p_{14} + p_{23}),$$
$$x = \tfrac{1}{\sqrt{2}}(p_{12} - p_{34}), \quad y = \tfrac{1}{\sqrt{2}}(p_{13} + p_{24}), \quad z = \tfrac{1}{\sqrt{2}}(p_{14} - p_{23}).$$

This is simultaneous rotation by $\pi/4$ in each of the coordinate planes spanned by the pairs $(p_{12}, p_{34})$, $(p_{13}, p_{24})$, and $(p_{14}, p_{23})$. In these new coordinates, the prime ideal (7.2) equals

$$\big\langle\, u^2 + v^2 + w^2 - \frac{1}{2}\,,\; x^2 + y^2 + z^2 - \frac{1}{2} \,\big\rangle.$$

This reveals that $\mathcal{G}_{2,4}$ is the direct product of two three-dimensional balls of radius $1/\sqrt{2}$. □

We next examine the case $d = 2$ and arbitrary $n$. The vectors $p$ in $\wedge_2 \mathbb{R}^n$ can be identified with skew-symmetric $n{\times}n$-matrices, and this brings us back to the orbitopes in Section 3.2.

**Corollary 7.2.** *The Grassmann orbitope $\mathcal{G}_{2,n}$ coincides with the skew-symmetric Schur-Horn orbitope of a skew-symmetric matrix $N \in \wedge_2\mathbb{R}^n$ having rank two and $\Lambda(N) = (1, 0, 0, \dots, 0)$.*

If $p$ is a real skew-symmetric matrix whose eigenvalues are $\pm i\tilde{\lambda}_1, \dots, \pm i\tilde{\lambda}_k$, where $i = \sqrt{-1}$, then the matrix $i \cdot p$ is Hermitian and its eigenvalues are the real numbers $\pm\tilde{\lambda}_1, \dots, \pm\tilde{\lambda}_k$. Recall that the operator $\mathcal{L}_k$ computes the $k$-th additive compound matrix of a given matrix.

**Theorem 7.3.** *Let $n \geq 5$ and $k = \lfloor n/2 \rfloor$. The Grassmann orbitope equals the spectrahedron*

$$\mathcal{G}_{2,n} \;=\; \big\{\, p \in \wedge_2\mathbb{R}^n \,:\, \mathrm{Id}_{\binom{n}{k}} - \mathcal{L}_k(i \cdot p) \,\succeq\, 0 \,\big\}. \tag{7.3}$$

*Its algebraic boundary $\partial_a\mathcal{G}_{2,n}$ is an irreducible hypersurface of degree $2^k$, defined by a factor of the determinant of the matrix $\mathrm{Id}_{\binom{n}{k}} - \mathcal{L}_k(i\cdot p)$. The proper faces of $\mathcal{G}_{2,n}$ are $SU(m)$-orbitopes for $1 \leq m \leq k$. Every face $F$ is associated with an even-dimensional subspace $V_F$ equipped with an orthogonal complex structure and the extreme points of $F$ correspond to complex lines in $V_F$.*

Everything in this theorem is also true for the small cases $n = 3, 4$, with the exception that the quartic hypersurface $\partial_a\mathcal{G}_{2,4}$ is not irreducible, as was seen in Example 7.1. In light of Theorem 3.11, the reducibility of $\mathcal{G}_{2,4}$ arises because the two-dimensional crosspolytope equals the square, which decomposes as a Minkowski sum of two line segments. This may also be seen as the stabilizer of a decomposable tensor in $SO(4)$ is $\pm I$, where $I$ is the identity matrix, and $SO(4)/\{\pm I\} \simeq SO(3) \times SO(3)$, so $\mathcal{G}_{2,4}$ is also an orbitope for $SO(3) \times SO(3)$.

*Proof.* We begin with the last statement about the face lattice of $\mathcal{G}_{2,n}$. This result is well-known in the theory of calibrations, where it is usually phrased as follows: *every face of the Grassmannian of two-planes in $\mathbb{R}^n$ consists of the complex lines in some $2m$-dimensional subspace of $\mathbb{R}^n$ under some orthogonal complex structure.* See [26, §1.1].

To derive the spectrahedral representation (7.3), we note that the eigenvalues of $\mathcal{L}_k(i \cdot p)$ are the sums of any $k$ distinct numbers of the eigenvalues of the skew-symmetric matrix $p$. These are $-\tilde{\lambda}_1, \dots -\tilde{\lambda}_k, \tilde{\lambda}_1, \dots, \tilde{\lambda}_k$ if $n$ is even and $-\tilde{\lambda}_1, \dots, -\tilde{\lambda}_k, 0, \tilde{\lambda}_1, \dots, \tilde{\lambda}_k$ if $n$ is odd. In light of Corollary 7.2, we can apply the results in Section 3.2 to conclude that $p$ lies in $\mathcal{G}_{2,n}$ if and only if $\pm\tilde{\lambda}_1 \pm \tilde{\lambda}_2 \pm \cdots \pm \tilde{\lambda}_k \leq 1$ for all choices of signs. In terms of polyhedral geometry, this condition means that the vector $\Lambda(p) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_k)$ lies in the crosspolytope $\Lambda(\mathcal{G}_{2,n})$. Since all $\binom{n}{k}$ eigenvalues of $\mathcal{L}_k(i \cdot p)$ are bounded above by the maximum of the $2^k$ special eigenvalues $\pm\tilde{\lambda}_1 \pm \tilde{\lambda}_2 \pm \cdots \pm \tilde{\lambda}_k$, we conclude that $p \in \mathcal{G}_{2,n}$ if and only if $\mathrm{Id}_{\binom{n}{k}} - \mathcal{L}_k(i\cdot p) \,\succeq\, 0$.

To compute the algebraic boundary $\partial_a\mathcal{G}_{2,n}$ we consider the expression

$$\prod_{\sigma \in \{-1,+1\}^k} (1 + \sigma_1\tilde{\lambda}_1 + \sigma_2\tilde{\lambda}_2 + \cdots + \sigma_k\tilde{\lambda}_k).$$

This is a symmetric polynomial of degree $2^{k-1}$ in the squared eigenvalues $\tilde{\lambda}_1^2, \tilde{\lambda}_2^2, \dots, \tilde{\lambda}_k^2$, and hence it can be written as a polynomial in the coefficients of the characteristic polynomial

$$\det\big(i \cdot p - x \cdot \mathrm{Id}_n\big) \;=\; x^{n \bmod 2} \cdot (x^2 - \tilde{\lambda}_1^2)(x^2 - \tilde{\lambda}_2^2) \cdots (x^2 - \tilde{\lambda}_k^2).$$

The resulting polynomial has degree $2^k$ in the entries $p_{ab}$ of the matrix $p$, and it vanishes on the boundary of the orbitope $\mathcal{G}_{2,n}$. It can be checked that it is irreducible for $n \geq 5$. □

**Example 7.4.** Let $n = 6$ and consider the characteristic polynomial of our Hermitian matrix:

$$\det\begin{pmatrix} -x & ip_{12} & ip_{13} & ip_{14} & ip_{15} & ip_{16} \\ -ip_{12} & -x & ip_{23} & ip_{24} & ip_{25} & ip_{26} \\ -ip_{13} & -ip_{23} & -x & ip_{34} & ip_{35} & ip_{36} \\ -ip_{14} & -ip_{24} & -ip_{34} & -x & ip_{45} & ip_{46} \\ -ip_{15} & -ip_{25} & -ip_{35} & ip_{45} & -x & ip_{56} \\ -ip_{16} & -ip_{26} & -ip_{36} & ip_{46} & -ip_{56} & -x \end{pmatrix} \begin{aligned} &= \quad x^6 + a_4x^4 + a_2x^2 + a_0 \\ &= \quad (x^2 - \tilde{\lambda}_1^2)(x^2 - \tilde{\lambda}_2^2)(x^2 - \tilde{\lambda}_3^2) \end{aligned}$$

The algebraic boundary of the Grassmann orbitope $\mathcal{G}_{2,6}$ is derived from the polynomial

$$\prod_{\sigma \in \{\pm 1\}^3} (1 + \sigma_1\tilde{\lambda}_1 + \sigma_2\tilde{\lambda}_2 + \sigma_3\tilde{\lambda}_3) \;=\; a_4^4 + 4a_4^3 - 8a_4^2a_2 + 16a_2^2 - 16a_4a_2 + 6a_4^2 + 64a_0 - 8a_2 + 4a_4 + 1\,.$$

We rewrite this expression in terms of the 15 unknowns $p_{ij}$ to get an irreducible polynomial of degree 8 with 10791 terms. This is the defining polynomial of the hypersurface $\partial_a\mathcal{G}_{2,6}$. For $n = 7$ we use the same polynomial in $a_0, a_2, a_4$ but now there is one more eigenvalue 0. The defining polynomial of $\partial_a\mathcal{G}_{2,7}$ has 44150 terms of degree 8 in the 21 matrix entries $p_{ij}$. □

We now come to the harder case $d = 3, n = 6$. The Grassmann orbitope $\mathcal{G}_{3,6}$ is a 20-dimensional convex body. Its facial structure was determined by Dadok and Harvey in [9], and independently by Morgan in [26]. We present their well-known results in our language.

**Theorem 7.5.** *The orbitope $\mathcal{G}_{3,6}$ has three classes of positive-dimensional exposed faces:*

(1) *one-dimensional faces from pairs of subspaces that satisfy the angle condition (7.4);*
(2) *three-dimensional faces that arise as $SO(3)$-orbitopes and these are round 3-balls;*
(3) *12-dimensional faces that are $SU(3)$-orbitopes, from the Lagrangian Grassmannian.*

Harvey and Morgan [17] extended these results to $\mathcal{G}_{3,7}$. We will here focus on $d = 3, n = 6$. Our first goal is to explain the faces described in Theorem 7.5, starting with the edges. Let $L$ and $L'$ be three-dimensional linear subspaces in $\mathbb{R}^6$ with corresponding unit length Plücker vectors $p$ and $p'$. We define $\theta_1(L, L')$ to be the minimum angle between any unit vector $v_1 \in L$ and any unit vector $w_1 \in L'$. Next, $\theta_2(L, L')$ is the minimum angle between two unit vectors $v_2 \in L$ and $w_2 \in L'$ such that $v_2 \perp v_1$ and $w_2 \perp w_1$. Finally, we define $\theta_3(L, L')$ to be the angle between two unit vectors $v_3 \in L$ and $w_3 \in L'$ such that $v_3 \perp \{v_1, v_2\}$ and $w_3 \perp \{w_1, w_2\}$. We refer to [26, Lemma 2.3] for the fact that $\theta_2$ and $\theta_3$ are well defined by this rule. The angle condition referred to in part (1) of Theorem 7.5 is the inequality

$$\theta_3(L, L') \;<\; \theta_1(L, L') + \theta_2(L, L'). \tag{7.4}$$

The convex hull of $p$ and $p'$ is an exposed edge of $\mathcal{G}_{3,6}$ if and only if the condition (7.4) holds.

The maximal facets in part (3) of Theorem 7.5 are known to geometers as *special Lagrangian facets*, and we represent them as orbitopes as follows. The *special unitary group*

$SU(3)$ consists of all complex 3×3-matrices $U$ with $\det(U) = 1$ and $UU^* = \mathrm{Id}_3$. If $A = \mathrm{re}(U)$ and $B = \mathrm{im}(U)$, so that $U = A + iB$, then we can identify $U$ with the real 6×6-matrix

$$\tilde{U} \;=\; \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Note that this matrix lies in $SO(6)$ if and only if $AA^T + BB^T = \mathrm{Id}_3$, $AB^T = BA^T$, and $\det(A+iB) = 1$. Hence the transformation $U \mapsto \tilde{U}$ realizes $SU(3)$ as a subgroup of $SO(6)$.

The orbit $SU(3) \cdot e_1 \wedge e_2 \wedge e_3$ is known as the *special Langrangian Grassmannian*. This is a five-dimensional real algebraic variety in the unit sphere in $\wedge_3 \mathbb{R}^6$. We call its convex hull

$$\mathcal{SL}_{3,6} \;=\; \mathrm{conv}(SU(3) \cdot e_1 \wedge e_2 \wedge e_3)$$

the *special Lagrangian orbitope*. It is obtained from the 20-dimensional Grassmann orbitope $\mathcal{G}_{3,6}$ by maximizing the linear function $p_{123} - p_{156} + p_{246} - p_{345}$, which takes value 1 on that face. The face $\mathcal{SL}_{3,6}$ is 12-dimensional because it satisfies the two affine equations

$$p_{123} - p_{156} + p_{246} - p_{345} \;=\; \mathrm{re}(A+iB) \;=\; 1 \quad \text{and} \quad p_{126} - p_{135} + p_{234} - p_{456} \;=\; \mathrm{im}(A+iB) \;=\; 0$$

plus the six independent linear equations that cut out the Langrangian Grassmannian:

$$p_{125} + p_{136} \;=\; p_{134} + p_{235} \;=\; p_{124} - p_{236} \;=\; p_{146} + p_{256} \;=\; p_{245} + p_{346} \;=\; p_{145} - p_{356} \;=\; 0.$$

The exposed faces of type (2) in Theorem 7.5 are not facets. Each of them is the intersection of two special Lagrangian facets. For example, consider the two linear functionals

$$\phi_+ \;=\; p_{123} - p_{156} + p_{246} - p_{345} \quad \text{and} \quad \phi_- \;=\; p_{123} - p_{156} - p_{246} + p_{345},$$

which were discussed in [26, §4.4]. Each of them supports a special Lagrangian facet. The intersection of these two 14-dimensional facets is a three-dimensional ball. Indeed, the linear functional $\frac{1}{2}(\phi_+ + \phi_-) = p_{123} - p_{156}$ is bounded above by 1 on $\mathcal{G}_{3,6}$, and the subset at which the value equals 1 is contained in the linear span of the six basis vectors $e_1 \wedge e_i \wedge e_j$ where $i, j \in \{2, 3, 5, 6\}$. In the intersection of this linear space with $\mathcal{G}_{3,6}$ we find the Grassmann orbitope $\mathcal{G}_{2,4}$ from Example 7.1, with our linear functional $p_{123} - p_{156}$ being represented by the scaled coordinate $\sqrt{2}x$. The face of $\mathcal{G}_{2,4}$ where $\sqrt{2}x$ attains its maximal value 1 is a 3-ball. This concludes our discussion of the census of faces given in Theorem 7.5.

A natural question that arises next is whether the Grassmann orbitope $\mathcal{G}_{3,6}$ or its dual body $\mathcal{G}^\circ_{3,6}$ can be represented as a spectrahedron. It turns out that the answer is negative.

**Theorem 7.6.** *The Grassmann coorbitope $\mathcal{G}^\circ_{3,6}$ has extreme points that are not exposed. The Grassmann orbitope $\mathcal{G}_{3,6}$ has edges that are not exposed. Neither of them is a spectrahedron.*

*Proof.* The first assertion was proved by Dadok and Harvey in [9, Theorem 7]. For the second assertion we proceed as follows. We apply the technique in [26, Lemma 2.2] and restrict to the linear subspace $V = \mathbb{R}\{e_{123}, e_{126}, e_{135}, e_{234}, e_{156}, e_{246}, e_{345}, e_{456}\}$ of $\wedge_3 \mathbb{R}^6$. The intersection of $\mathcal{G}_{3,6} \cap V$ is the $SO(2) \times SO(2) \times SO(2)$-orbitope of $e_{123}$, where the action is by unitary diagonal 3×3-matrices, while the intersection $\mathcal{SL}_{3,6} \cap V$ is the $SO(2) \times SO(2)$-orbitope of $e_{123}$, where the action is by special unitary diagonal 3×3-matrices, as described in [26, §4.3].

We claim that the $SO(2) \times SO(2)$-orbitope $\mathcal{SL}_{3,6} \cap V$ is 2-neighborly. This can be seen by examining the bivariate trigonometric polynomials of the form

$$\begin{aligned} f \;=\; & \phantom{+}\, x_{123} \cdot \cos(\alpha)\cos(\beta)\cos(-\alpha-\beta) \;+\; x_{126} \cdot \cos(\alpha)\cos(\beta)\sin(-\alpha-\beta) \\ & - x_{135} \cdot \cos(\alpha)\sin(\beta)\cos(-\alpha-\beta) \;+\; x_{234} \cdot \sin(\alpha)\cos(\beta)\sin(-\alpha-\beta) \\ & + x_{156} \cdot \cos(\alpha)\sin(\beta)\sin(-\alpha-\beta) \;-\; x_{246} \cdot \sin(\alpha)\cos(\beta)\sin(-\alpha-\beta) \\ & + x_{345} \cdot \sin(\alpha)\sin(\beta)\cos(-\alpha-\beta) \;-\; x_{456} \cdot \sin(\alpha)\sin(\beta)\sin(-\alpha-\beta). \end{aligned}$$

Indeed, for any choice of $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$, we can find eight real coefficients $x_{ijk}$ such that $f$ and its derivatives vanish at $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ but $f$ is strictly positive elsewhere.

The results in [26] imply that every exposed edge of $\mathcal{G}_{3,6} \cap V$ is also an exposed edge of $\mathcal{G}_{3,6}$. We believe that Morgan's technique can be adapted to show that *every* exposed edge of $\mathcal{SL}_{3,6} \cap V$ is an exposed edge of $\mathcal{SL}_{3,6}$. For the purpose of proving Theorem 7.6, however, we only need to identify *one* exposed edge of $\mathcal{SL}_{3,6} \cap V$ that is an exposed edge of $\mathcal{SL}_{3,6}$.

The claim that such exposed edges exist can be derived from [9, Theorem 12 (i)]. With the help of Philipp Rostalski, we also obtained a computational proof of that claim. This was done as follows. We selected various random choices of points $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ in $\mathbb{R}^2$. Each choice specifies two three-dimensional subspaces $L$ and $L'$ of $\mathbb{R}^6$ for which equality holds in the angle condition (7.4). The corresponding line segment is not an exposed edge of $\mathcal{G}_{3,6}$.

To show that the line segment between $L$ and $L'$ is exposed in $\mathcal{SL}_{3,6}$, we use a technique from semidefinite programming. First we compute the eight coefficients $x_{ijk}$ of the supporting function $f$ as above. This gives a linear function $\sum x_{ijk} p_{ijk}$ on $\wedge_3 \mathbb{R}^6$. We then run a first-order Lasserre relaxation (cf. [21]) to minimize this linear function subject to the linear and quadratic constraints that cut out the special Lagrangian Grassmannian. The optimal value is zero, the optimal Lasserre moment matrix has rank two, and its image in $\wedge_3 \mathbb{R}^6$ lies in the relative interior of the line segment between $L$ and $L'$. We then re-optimize for various perturbations of the linear function $\sum x_{ijk} p_{ijk}$. The output of each run is a rank one moment matrix which certifies either $L$ or $L'$ as optimal solution of the optimization problem. This proves that the face of $\mathcal{SL}_{3,6}$ exposed by $\sum x_{ijk} p_{ijk}$ is the line segment between $L$ and $L'$. $\square$

Theorem 7.6 shows that Grassmann orbitopes are generally not spectrahedra. We do not know whether they are linear projections of spectrahedra.

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