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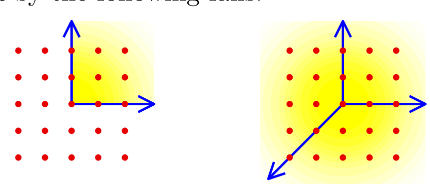
Toric varieties, by David A. Cox, John B. Little, and Henry K. Schenck, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, xxiv+841 pp., \$95, ISBN: 978-0-8218-4819-7

Algebraic Geometry is a deep subject of remarkable power that originated in the study of concrete objects, as it is concerned with the geometry of solutions to polynomial equations. It is this interplay between theoretical abstraction and tangible examples that gives the subject relevance and a reach into other areas of mathematics and its applications. There is no better subdiscipline of algebraic geometry than that of toric varieties for observing the application of its ideas and methods on a wide variety of explicit geometric objects.

While toric varieties have been around as long as algebraic geometry—the familiar Cartesian plane and the projective plane are both toric varieties as is the parabola $y = x^2$—the subject of toric varieties began with the foundational paper of Demazure [5]. Toric varieties arose as a byproduct of his main interest, groups of rational automorphisms of k^n (k a field) that contain a split torus, $(k^*)^n$. These groups turn out to be automorphism groups of smooth toric varieties, which Demazure defined and classified as schemes over $\mathbb{Z}$. These are obtained by adding certain points at infinity to a split torus, and the classification involved objects from geometric combinatorics called *éventails* (fans *en anglais*) encoding the added points. Fans are certain collections of cones in $\mathbb{Z}^n$, with each cone corresponding to an orbit of $(k^*)^n$ in the toric variety, and this correspondence is contravariant. For example, the Cartesian k^2 and projective $\mathbb{P}^2(k)$ planes are compactifications of the torus $(k^*)^2$ encoded by the following fans.



In both, the origin corresponds to the dense torus $(k^*)^2$, the horizontal and vertical rays to the (open) x- and y-axes, respectively, and the positive quadrant to the point $x = y = 0$. In the fan on the right for $\mathbb{P}^2(k)$, the diagonal ray corresponds to the line at infinity with the two adjacent regions to the points at infinity on the x- and y-axes. These are smooth because the first vectors on the rays of each cone (origin, ray, or region) are parts of a basis for $\mathbb{Z}^2$.

These ideas were revisited by Kempf, Knudsen, Mumford and Saint-Donat [11] and Miyake and Oda [14]. Both sets of authors considered normal (not just smooth) varieties X over a field (hereafter the complex numbers, $\mathbb{C}$) containing a torus

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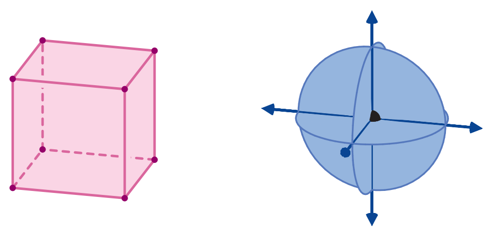
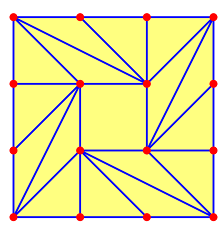
$(\mathbb{C}^*)^n$ on which the action of the torus on itself extends to X. As before, these normal equivariant compactifications (toric varieties) are classified by arbitrary fans Σ in $\mathbb{Z}^n$ (not just smooth fans). A consequence of this classification was a rich class of algebraic varieties—toric varieties—whose structure could be understood completely in terms of the geometry and combinatorics of the corresponding fans. For example, if each cone in the fan σ is simplicial (the first vectors on each of its rays are linearly independent), then the associated toric variety X_Σ is an orbifold. Also, X_Σ is compact if the union of the cones in Σ is $\mathbb{Z}^n$.

A toric variety X_Σ can be embedded into projective space if and only if its fan Σ is the domain of a piecewise linear convex function in which each cone is a region of linearity with a different linear function for each maximal cone. A given projective embedding corresponds to a polytope P with integer vertices (lattice polytope) whose normal fan is Σ. That is, the cones of Σ are the outer normal vectors to the faces of P. Consider the normal fan to the cube. The rays are normal to the facets of the cube, the sectors between the rays are normal to the edges, and the orthants are normal to the vertices.



This determination of projective embeddings shows that all toric surfaces are quasi-projective and gives many easy examples of non-projective algebraic varieties, as it is a very strong restriction on a fan to be the normal fan to a lattice polytope. For example, consider the following decomposition of the 3×3 square in $\mathbb{Z}^2$.

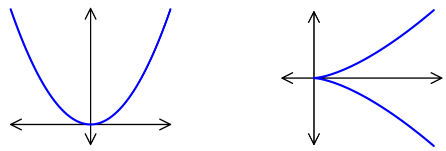


There is no piecewise linear convex function on the square whose maximal regions of linearity are exactly the triangles and the central square. If there were, then adding an affine function, we may assume that it is constant on the central square, and then the convex function must be decreasing moving clockwise from each vertex, which is impossible outside of Escher woodcuts. This is the intersection at $z = 1$ of a fan in $\mathbb{Z}^3$ that gives a non-projective three-dimensional toric variety.

The usefulness of toric varieties goes beyond the plethora of accessible examples afforded by this classification. An important variant are varieties in affine or projective space that are parameterized by monomials, which are also called toric varieties. An integer vector $\alpha \in \mathbb{Z}^n$ is the exponent of a Laurent monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, for $x \in (\mathbb{C}^*)^n$. A finite collection $\mathcal{A} \subset \mathbb{Z}^n$ of integer vectors gives a map $\varphi_\mathcal{A} \colon (\mathbb{C}^*)^n \to \mathbb{C}^\mathcal{A}$,

$$\varphi_\mathcal{A}(x) \;=\; (x^\alpha \mid \alpha \in \mathcal{A})\,. \tag{1}$$

(We use elements of $\mathcal{A}$ as indices.) The closure of the image of $\varphi_{\mathcal{A}}$ in either affine space $\mathbb{C}^{\mathcal{A}}$ or projective space $\mathbb{P}(\mathbb{C}^{\mathcal{A}})$ is the toric variety $X_{\mathcal{A}}$. This is an equivariant compactification of the torus $(\mathbb{C}^*)^n$ as before. For example, if $\mathcal{A} = \{1, 2\}$ so that $\varphi_{\mathcal{A}}(t) = (t, t^2)$ then $X_{\mathcal{A}}$ is the parabola $y = x^2$, and if $\mathcal{A} = \{2, 3\}$ so that $\varphi_{\mathcal{A}}(t) = (t^2, t^3)$ then $X_{\mathcal{A}}$ is the semicubical parabola $y^2 = x^3$ (also called the cuspidal cubic).



This second notion of a toric variety via monomial parameterization is distinct from the first notion of toric varieties associated to fans in $\mathbb{Z}^n$. Indeed, the toric variety $X_{\mathcal{A}}$ comes with an embedding into affine or projective space, while toric varieties associated to fans have no preferred embedding (or none at all). More fundamentally, $X_{\mathcal{A}}$ need not be normal—due to its cusp, the semicubical parabola is not a normal variety.

Monomial parameterizations are the reason that toric varieties figure prominently in the study of subvarieties of $(\mathbb{C}^*)^n$ defined by Laurent polynomials

$$f_1(x) \;=\; f_2(x) \;=\; \cdots \;=\; f_r(x) \;=\; 0\,. \tag{2}$$

A Laurent polynomial f with support $\mathcal{A}$ is a linear combination

$$f(x) \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha \qquad c_\alpha \in \mathbb{C}^*\,, \tag{3}$$

of monomials from $\mathcal{A}$. Its set of zeros $(f(x) = 0)$ is either a nonlinear subset of $(\mathbb{C}^*)^n$ or (via the map $\varphi_{\mathcal{A}}$) a section of the toric variety $X_{\mathcal{A}}$ by the hyperplane

$$0 \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha z_\alpha\,.$$

where $\{z_\alpha \mid \alpha \in \mathcal{A}\}$ are coordinates on $\mathbb{P}(\mathbb{C}^{\mathcal{A}})$.

A useful invariant of a Laurent polynomial (3) is its Newton polytope, which is the convex hull of the set $\mathcal{A}$ of its exponent vectors, and which controls much of the geometry of the associated toric variety, $X_{\mathcal{A}}$. Bernstein, Khovanskii, and Kushnirenko showed that Newton polytopes of the polynomials f_i in (2) control the geometry of the subvariety when the polynomials were generic. For example, Bernstein [1] showed that when $r = n$ so that (2) is a system of polynomials with finitely many solutions, the number of solutions in $(\mathbb{C}^*)^n$ is the mixed volume of the Newton polytopes, and Khovanskii expressed the Euler characteristic of a hypersurface in $(\mathbb{C}^*)^n$ defined by a polynomial f in terms of the volume of the Newton polytope of f. These results may be obtained using the toric variety $X_{\mathcal{A}}$. Danilov wrote the first survey on toric varieties [4], and it contained further results linking toric varieties to polytopes such as using the Riemann-Roch Theorem to enumerate lattice points in polytopes. We are indebted to Miles Reid, who coined the elegant term ‘toric variety’ when translating Danilov’s survey into English.

A milestone in this early history was Stanley’s proof of the McMullen conjecture [15], using the hard Lefschetz Theorem in the cohomology of an associated toric variety. The simplest enumerative invariants of a d-dimensional polytope P are its face numbers $f_0, \ldots, f_d$ where f_i counts the number of i-dimensional faces of P. McMullen had given a list of inequalities on the f_i when P is simplicial (all faces are simplices) and conjectured that they were necessary and sufficient for there to be a polytope with those face numbers. Billera and Lee proved sufficiency using a construction. Stanley showed necessity, first observing that P may be assumed to be a lattice polytope. The cones over the faces of P form a complete simplicial fan Σ in $\mathbb{Z}^d$ that is the normal fan of the polytope dual to P. This implies that the associated toric variety X_Σ is a projective orbifold, and Stanley showed that the McMullen inequalities follow from Poincaré duality and the hard Lefschetz Theorem, both of which hold for X_Σ.

By the mid 1990's, toric varieties had entered the consciousness if not the toolkit of many in algebraic geometry and related fields. One reason was noted by Fulton, who remarked that "toric varieties have proved a remarkably fertile testing ground for general theories", including resolution of singularities, classification of higher dimensional varieties, Hodge theory, Riemann-Roch theorems, intersection theory, general cohomology theories, and vanishing theorems. Another was the publication of several influential books on toric varieties including the mature books of Oda [13] in 1988 and Fulton [8] in 1993 (in which the above quotation appeared). These books were written for algebraic geometers, and with their publication, toric varieties became a path into the subject.

Other books of that period included that of Ewald [7] in 1996, which treated both toric varieties and geometric combinatorics and was aimed at those outside of algebraic geometry. The seminal book of Gel′fand, Kapranov, and Zelevinsky [9] in 1994 systematically treated the nonnormal affine toric varieties given by monomial parameterization (1), and introduced toric degenerations and secondary fans. Sturmfels's lecture notes [16] of 1996 also had a deep influence, emphasizing links to commutative algebra and geometric combinatorics through toric ideals (ideals of toric varieties given by monomial parameterization (1)), their relation to toric degenerations and Gröbner bases, and their role in some applications including integer programming and statistical inference.

The former and latter books championed different versions of toric varieties—normal toric varieties from fans for algebraic geometry and affine toric varieties parameterized by monomials for combinatorics, commutative algebra, and applications. Alluding to the authors, these have been cheekily referred to as "Fultonian" and "Sturmfellian" toric varieties, respectively.

Since that period, ideas and techniques from toric varieties have penetrated deeply into algebraic geometry, commutative algebra, combinatorics, and related fields. While this is due in part to those books, it has also been driven by new ideas and fields that have since arisen. For example, it was realized (almost simultaneously by several authors) that toric varieties are symplectic or geometric invariant theory quotients of affine space by a torus. Among the consequences is the existence of a universal homogeneous coordinate ring of a toric variety, called its Cox ring. Toric varieties appeared in work studying mirror symmetry from theoretical physics, which then introduced new ideas and methods into the subject. The new field of tropical geometry is intimately connected to the world of toric varieties, as its objects are combinatorial shadows of subvarieties of a torus $(\mathbb{K}^*)^n$, where $\mathbb{K}$ is typically a valued field.

Just as toric varieties have proven fruitful for the development of pure mathematics as fundamental objects and for testing general theories, they are playing the same role in many emerging applications of algebraic geometry. This is seen not only for integer programming in [16], but also for solutions to systems of polynomial equations [3], geometric modeling [10], coding theory [12], and algebraic statistics [6].

Their phenomenal development in the past two decades created a need for a comprehensive treatise on toric varieties. This need has been fulfilled by the encyclopedic book under review, *Toric varieties*, by Cox, Little, and Schenck.

This book begins with a careful development of three distinct types of toric varieties, treating in turn affine toric varieties parameterized by monomials, projective toric varieties corresponding to lattice polytopes, and normal toric varieties corresponding to fans. The remaining six chapters of Part I treat the fundamental algebraic geometry of toric varieties, including divisors, line bundles and sheaves, the Cox ring, and sheaf cohomology. Part II is populated with more advanced topics on toric varieties, including Riemann-Roch theorems, geometric invariant theory, and the secondary fan. Appendices on the history of toric varieties, computational methods for toric varieties, and spectral sequences round out the text.

While its sheer size (over 800 pages) is intimidating, this book is best taken in manageable pieces. For example, the first three chapters and a selection of the later material make an excellent introductory course for either algebraic geometers or those with a more varied background. The book is accessible and self-contained, presenting significant background material. It is also a source for most things toric and consequently is a valuable reference. With its engaging and inviting writing, this will surely join other books (such as [2] by Cox, Little, and O'Shea) as an enduring classic that will occupy the shelves of those working in algebraic geometry, related fields, and applications.

References

1. D. N. Bernstein, *The number of roots of a system of equations*, Funkcional. Anal. i Priložen. **9** (1975), no. 3, 1–4. MR 0435072 (55 #8034)
2. David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992, An introduction to computational algebraic geometry and commutative algebra. MR 1189133 (93j:13031)
3. ———, *Using algebraic geometry*, Graduate Texts in Mathematics, vol. 185, Springer-Verlag, New York, 1998. MR 1639811 (99h:13033)
4. V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247. MR 495499 (80g:14001)
5. Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588. MR 0284446 (44 #1672)
6. Mathias Drton, Bernd Sturmfels, and Seth Sullivant, *Lectures on algebraic statistics*, Oberwolfach Seminars, vol. 39, Birkhäuser Verlag, Basel, 2009. MR 2723140 (2012d:62004)
7. Günter Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996. MR 1418400 (97i:52012)
8. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037 (94g:14028)
9. I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. MR 1264417 (95e:14045)
10. Ron Goldman and Rimvydas Krasauskas (eds.), *Topics in algebraic geometry and geometric modeling*, Contemporary Mathematics, vol. 334, American Mathematical Society, Providence, RI, 2003, Papers from the Workshop on Algebraic Geometry and Geometric Modeling held at Vilnius University, Vilnius, July 29–August 2, 2002. MR 2046835 (2004j:00026)

11. G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973. MR 0335518 (49 #299)
12. Edgar Martínez-Moro, Carlos Munuera, and Diego Ruano (eds.), *Advances in algebraic geometry codes*, Series on Coding Theory and Cryptology, vol. 5, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. MR 2516523 (2010a:94001)
13. Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese. MR 922894 (88m:14038)
14. Tadao Oda and Katsuya Miyake, *Almost homogeneous algebraic varieties under algebraic torus action*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 373–381. MR 0379501 (52 #406)
15. Richard P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238. MR 563925 (81f:52014)
16. Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR 1363949 (97b:13034)

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