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EXPERIMENTATION AND CONJECTURES IN THE REAL SCHUBERT CALCULUS FOR FLAG MANIFOLDS

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Abstract. The Shapiro conjecture in the real Schubert calculus, while likely true for Grassmannians, fails to hold for flag manifolds, but in a very interesting way. We give a refinement of the Shapiro conjecture for flag manifolds and present massive computational experimentation in support of this refined conjecture. We also prove the conjecture in some special cases using discriminants and establish relationships between different cases of the conjecture.

Introduction

The Shapiro conjecture for Grassmannians [24, 18] has driven progress in enumerative real algebraic geometry [27], which is the study of real solutions to geometric problems. It conjectures that a (zero-dimensional) intersection of Schubert subvarieties of a Grassmannian consists entirely of real points—if the Schubert subvarieties are given by flags osculating a real rational normal curve. This particular Schubert intersection problem is quite natural; it can be interpreted in terms of real linear series on $\mathbb{P}^1$ with prescribed (real) ramification [1, 2], real rational curves in $\mathbb{P}^n$ with real flexes [11], linear systems theory [16], and the Bethe ansatz and Fuchsian equations [14]. The Shapiro conjecture has implications for all these areas. Massive computational evidence [24, 29] as well as its proof by Eremenko and Gabrielov for Grassmannians of codimension 2 subspaces [4] give compelling evidence for its validity. A local version, that it holds when the Schubert varieties are special (a technical term) and when the points of osculation are sufficiently clustered [23], showed that the special Schubert calculus is fully real (such geometric problems can have all their solutions real). Vakil later used other methods to show that the general Schubert calculus on the Grassmannian is fully real. [28]

The original Shapiro conjecture stated that such an intersection of Schubert varieties in a *flag manifold* would consist entirely of real points. Unfortunately, this conjecture fails for the first non-trivial enumerative problem on a non-Grassmannian flag manifold, but in a very interesting way. Failure for flag manifolds was first noted in [24, §5] and a more symmetric counterexample was found in [25], where computer experimentation suggested that the conjecture would hold if the points where the flags osculated the rational normal curve satisfied a certain non-crossing condition. Further experimentation led to a precise formulation of this refined non-crossing conjecture in [27]. That conjecture was only valid

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for two- and three- step flag manifolds, and the further experimentation reported here leads to versions (Conjectures 2.2 and 3.8) for all flag manifolds in which the points of osculation satisfy a monotonicity condition.

We have systematically investigated the Shapiro conjecture for flag manifolds to gain a deeper understanding both of its failure and of our refinement. This investigation includes 15.76 gigahertz-years of computer experimentation, theorems relating our conjecture for different enumerative problems, and its proof in some cases using discriminants. Recently, our conjecture was proven by Eremenko, Gabrielov, Shapiro, and Vainshtein [5] for manifolds of flags consisting of a codimension 2 plane lying on a hyperplane. Our experimentation also uncovered some new and interesting phenomena in the Schubert calculus of a flag manifold, and it included substantial computation in support of the Shapiro conjecture on the Grassmannians $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(4,8)$.

Our conjecture is concerned with a subclass of Schubert intersection problems. Here is one open instance of this conjecture, expressed as a system of polynomials in local coordinates for the variety of flags $E_2 \subset E_3$ in 5-space, where $\dim E_i = i$. Let $t, x_1, \ldots, x_8$ be indeterminates, and consider the polynomials

$$f(t;x) \quad := \quad \det \begin{bmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ t^4 & t^3 & t^2 & t & 1 \\ 4t^3 & 3t^2 & 2t & 1 & 0 \\ 12t^2 & 6t & 2 & 0 & 0 \end{bmatrix}, \qquad \text{and}$$

$$g(t;x) \quad := \quad \det \begin{bmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 0 & 0 & 1 & x_7 & x_8 \\ t^4 & t^3 & t^2 & t & 1 \\ 4t^3 & 3t^2 & 2t & 1 & 0 \end{bmatrix}.$$

Conjecture A. *Let $t_1 < t_2 < \cdots < t_8$ be real numbers. Then the polynomial system*

$$\begin{aligned} f(t_1;x) = f(t_2;x) = f(t_3;x) = f(t_4;x) \quad &= \quad 0, \quad \textit{and} \\ g(t_5;x) = g(t_6;x) = g(t_7;x) = g(t_8;x) \quad &= \quad 0 \end{aligned}$$

has 12 solutions, and all of them are real.

Evaluating the polynomial f at points t_i preceeding the points at which the polynomial g is evaluated is the monotonicity condition. If we had switched the order of t_4 and t_5,

$$t_1 < t_2 < t_3 < \quad t_5 < t_4 \quad < t_6 < t_7 < t_8 ,$$

then this would not be monotone. We computed 400,000 instances of this polynomial system at different choices of points $t_1 < \cdots < t_8$ (which were monotone), and each had 12 real solutions. In contrast, there were many non-monotone choices of points for which not all solutions were real, and the minimum number of real solutions that we observe seems to depend on the combinatorics of the evaluation. For example, the system with interlaced points t_i

$$f(-8;x) = g(-4;x) = f(-2;x) = g(-1;x) = f(1;x) = g(2;x) = f(4;x) = g(8;x) = 0$$

has 12 solutions, *none* of which are real. This investigation is summarized in Table 1.

This paper is organized as follows. In Section 1, we provide background material on flag manifolds, state the Shapiro Conjecture, and give a geometrically vivid example of its failure. In Section 2, we give the results of our experimentation, stating our conjectures and describing some interesting phenomena that we have observed in our data. The discussion in Section 3 contains theorems about our conjectures, a generalization of our main conjecture, and proofs of it in some cases using discriminants. Finally, in Section 4 we describe our methods, explain our experimentation and give a brief guide to our data, all of which and much more is tabulated and available on line at `www.math.tamu.edu/~sottile/pages/Flags/`.

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1. Background

1.1. **Basics on flag manifolds.** Let $\alpha_1 < \cdots < \alpha_k < n$ be positive intgegers, and set $\alpha := \{\alpha_1 < \cdots < \alpha_k\}$. Let $\mathbb{F}\ell(\alpha;n)$ be the manifold of flags in $\mathbb{C}^n$ of *type* α,

$$\mathbb{F}\ell(\alpha;n) \;:=\; \{E_\bullet = E_{\alpha_1} \subset E_{\alpha_2} \subset \cdots \subset E_{\alpha_k} \subset \mathbb{C}^n \mid \dim E_{\alpha_i} = \alpha_i\}\,.$$

If we set $\alpha_0 := 0$, then this algebraic manifold has dimension

$$\dim(\alpha) \;:=\; \sum_{i=1}^{k} (n-\alpha_i)(\alpha_i - \alpha_{i-1})\,.$$

Complete flags in $\mathbb{C}^n$ have type $1 < 2 < \cdots < n{-}1$.

Define $W^\alpha \subset S_n$ to be the set of permutations with descents in α,

$$W^\alpha \;:=\; \{w \in S_n \mid i \notin \{\alpha_1, \ldots, \alpha_k\} \Rightarrow w(i) < w(i+1)\}\,.$$

We often write permutations as a sequence of their values, omitting commas if possible. Thus $(1,3,2,4,5) = 13245$ and 341526 are permutations in S_5 and S_6, respectively. Since a permutation $w \in W^\alpha$ is determined by its values before its last descent, we need only write its first α_k values. Thus $132546 \in W^{\{2,4\}}$ may be written 1325. Lastly, we write σ_i for the simple transposition $(i, i{+}1)$.

The positions of flags $E_\bullet$ of type α relative to a fixed complete flag $F_\bullet$ stratify $\mathbb{F}\ell(\alpha;n)$ into *Schubert cells*. The closure of a Schubert cell is a *Schubert variety*. Permutations $w \in W^\alpha$ index Schubert cells $X_w^\circ F_\bullet$ and Schubert varieties $X_w F_\bullet$ of $\mathbb{F}\ell(\alpha;n)$. More precisely, if we set $r_w(i,j) := |\{l \le i \mid j + w(l) > n\}|$, then

$$\begin{aligned} X_w^\circ F_\bullet &= \{E_\bullet \mid \dim E_{\alpha_i} \cap F_j = r_w(\alpha_i, j),\ i = 1,\ldots,k,\ j = 1,\ldots,n\}, \quad \text{and} \\ X_w F_\bullet &= \{E_\bullet \mid \dim E_{\alpha_i} \cap F_j \ge r_w(\alpha_i, j),\ i = 1,\ldots,k,\ j = 1,\ldots,n\}\,. \end{aligned} \tag{1.1}$$

Flags $E_\bullet$ in $X_w^\circ F_\bullet$ have *position w relative to $F_\bullet$*. We will refer to a permutation $w \in W^\alpha$ as a *Schubert condition* on flags of type α. The Schubert subvariety $X_w F_\bullet$ is irreducible with codimension $\ell(w) := |\{i < j \mid w(i) > w(j)\}|$ in $\mathbb{F}\ell(\alpha; n)$.

Schubert cells are affine spaces with $X_w^\circ F_\bullet \simeq \mathbb{C}^{\dim(\alpha)-\ell(w)}$. We introduce a convenient set of coordinates for Schubert cells. Let $\mathcal{M}_w$ be the set of $\alpha_k \times n$ matrices, some of whose entries $x_{i,j}$ are fixed: $x_{i,w(i)} = 1$ for $i = 1, \ldots, \alpha_k$ and $x_{i,j} = 0$ if

$$j < w(i) \ \text{ or } \ w^{-1}(j) < i \ \text{ or } \ \alpha_l < i < w^{-1}(j) \leq \alpha_{l+1} \text{ for some } l,$$

and whose remaining $\dim(\alpha) - \ell(w)$ entries give coordinates for $\mathcal{M}_w$. For example, if $n = 8$, $\alpha = (2, 3, 6)$, and $w = 25\,3\,167$, then $\mathcal{M}_w$ consists of matrices of the form

$$\begin{pmatrix} 0 & 1 & x_{13} & x_{14} & 0 & x_{16} & x_{17} & x_{18} \\ 0 & 0 & 0 & 0 & 1 & x_{26} & x_{27} & x_{28} \\ 0 & 0 & 1 & x_{34} & 0 & x_{36} & x_{37} & x_{38} \\ 1 & 0 & 0 & x_{44} & 0 & 0 & 0 & x_{48} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_{68} \end{pmatrix} .$$

The relation of $\mathcal{M}_w$ to the Schubert cell $X_w^\circ F_\bullet$ is as follows. Given a complete flag $F_\bullet$, choose an ordered basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ corresponding to the columns of matrices in $\mathcal{M}_w$ such that F_i is the linear span of the last i basis vectors, $e_{n+1-i}, \ldots, e_{n-1}, e_n$. Given a matrix $M \in \mathcal{M}_w$, set E_{α_i} to be the row space of the first α_i rows of M. Then the flag $E_\bullet$ has type α and lies in the Schubert cell $X_w^\circ F_\bullet$, every flag $E_\bullet \in X_w^\circ F_\bullet$ arises in this way, and the association $M \mapsto E_\bullet$ is an algebraic bijection between $\mathcal{M}_w$ and $X_w^\circ F_\bullet$. This is a flagged version of echelon forms. See [7] for details and proofs.

Let ι be the identity permutation. Then $\mathcal{M}_\iota$ provides local coordinates for $\mathbb{F}\ell(\alpha; n)$ in which the equations for a Schubert variety are easy to describe. Note that

$$\dim(E_{\alpha_i} \cap F_j) \ \geq \ r \quad \Longleftrightarrow \quad \operatorname{rank}(A) \ \leq \ \alpha_i + j - r\,,$$

where the matrix A is formed by stacking the first α_i rows of $\mathcal{M}_\iota$ on top of a $j \times n$ matrix with row span F_j. Algebraically, this rank condition is the vanishing of all minors of A of size $1+\alpha_i+j-r$. The polynomials f and g of Example A from the Introduction arise in this way. There $\alpha = \{2, 3\}$ and $\mathcal{M}_\iota$ is the matrix of variables in the definition of g.

Suppose that β is a subsequence of α. Then $W^\beta \subset W^\alpha$. Simply forgetting the components of a flag $E_\bullet \in \mathbb{F}\ell(\alpha; n)$ that do not have dimensions in the sequence β gives a flag in $\mathbb{F}\ell(\beta; n)$. This defines a map

$$\pi \ : \ \mathbb{F}\ell(\alpha; n) \ \longrightarrow \ \mathbb{F}\ell(\beta; n)$$

whose fibres are (products of) flag manifolds. The inverse image of a Schubert variety $X_w F_\bullet$ of $\mathbb{F}\ell(\beta; n)$ is the Schubert variety $X_w F_\bullet$ of $\mathbb{F}\ell(\alpha; n)$.

When $\beta = \{b\}$ is a singleton, $\mathbb{F}\ell(\beta; n)$ is the Grassmannian of b-planes in $\mathbb{C}^n$, written $\mathrm{Gr}(b, n)$. Non-identity permutations in W^β have a unique descent at b. A permutation w with a unique descent is *Grassmannian* as the associated Schubert variety $X_w F_\bullet$ (a *Grassmannian Schubert variety*) is the inverse image of a Schubert variety in a Grassmannian.

1.2. **The Shapiro Conjecture.** A list $(w_1, \ldots, w_m)$ of permutations in W^α is called a *Schubert problem* if $\ell(w_1) + \cdots + \ell(w_m) = \dim(\alpha)$. Given such a list and complete flags $F^1_\bullet, \ldots, F^m_\bullet$, consider the Schubert intersection

$$X_{w_1}F^1_\bullet \cap \cdots \cap X_{w_m}F^m_\bullet \,. \tag{1.2}$$

When the flags $F^i_\bullet$ are in general position, this intersection is zero-dimensional (in fact transverse by the Kleiman-Bertini theorem [12]), and it equals the intersection of the corresponding Schubert cells. In that case, the intersection (1.2) consists of those flags $E_\bullet$ of type α which have position w_i relative to $F^i_\bullet$, for each $i = 1, \ldots, m$. We call these *solutions* to the Schubert intersection problem (1.2). The number of solutions does not depend on the choice of flags (as long as the intersection is transverse) and we call this number the *degree* of the Schubert problem. This degree may be computed, for example, in the cohomology ring of the flag manifold $\mathbb{F}\ell(\alpha; n)$.

The Shapiro conjecture concerns the following variant of this classical enumerative geometric problem: Which *real* flags $E_\bullet$ have given position w_i relative to *real* flags $F^i_\bullet$, for each $i = 1, \ldots, m$? In the Shapiro conjecture, the flags $F^i_\bullet$ are not general real flags, but rather flags osculating a rational normal curve. Let $\gamma\colon \mathbb{C} \to \mathbb{C}^n$ be the rational normal curve, $\gamma(t) := (1, t, t^2, \ldots, t^{n-1})$ written with respect to the ordered basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ given above. The *osculating flag* $F_\bullet(t)$ of subspaces to γ at the point $\gamma(t)$ is the flag whose i-dimensional component is

$$F_i(t) \;:=\; \mathrm{span}\{\gamma(t), \gamma'(t), \ldots, \gamma^{(i-1)}(t)\}\,.$$

When $t = \infty$, the subspace $F_i(\infty)$ is spanned by $\{e_{n+1-i}, \ldots, e_n\}$ and $F_\bullet(\infty)$ is the flag used to describe the coordinates $\mathcal{M}_w$. If we consider this projectively, $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^{n-1}$ is the rational normal curve and $F_\bullet(t)$ is the flag of subspaces osculating γ at $\gamma(t)$.

Conjecture 1.3 (B. Shapiro and M. Shapiro)**.** *Suppose that $(w_1, \ldots, w_m)$ is a Schubert problem for flags of type α. If the flags $F^1_\bullet, \ldots, F^m_\bullet$ osculate the rational normal curve at distinct real points, then the intersection* 1.2 *is transverse and consists only of real points.*

The Shapiro conjecture is concerned with intersections of the form

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,, \tag{1.4}$$

where we write $X_w(t)$ for $X_wF_\bullet(t)$. This intersection is an *instance* of the Shapiro conjecture for the Schubert problem $(w_1, \ldots, w_m)$ at the points $(t_1, \ldots, t_m)$.

Conjecture 1.3 dates from around 1995. Experimental evidence of its validity for Grassmannians was first found in [16, 21]. This led to a systematic investigation on Grassmannians, both experimentally and theoretically in [24]. There, the conjecture was proven using discriminants for several (rather small) Schubert problems and relationships between the conjecture for different Schubert problems were established. (See also Theorem 2.8 of [11].) For example, if the Shapiro conjecture holds on a Grassmannian for the Schubert problem consisting only of codimension 1 (simple) conditions, then it holds for all Schubert problems on that Grassmannian and on all smaller Grassmannians, if we drop the claim of transversality. More recently, Eremenko and Gabrielov proved the conjecture for any Schubert problem on a Grassmannian of codimension 2-planes [4]. Their result is appealingly interpreted as a rational function all of whose critical points are real must be real.

The original conjecture was for flag manifolds, but a counterexample was found and reported in [24]. Subsequent experimentation refined this counterexample, and has suggested a reformulation of the original conjecture. We study this refined conjecture and report on massive computer experimentation (15.76 gigahertz-years) undertaken in 2003 and 2004 at the University of Massachusetts at Amherst, at the MSRI in 2004, and some at Texas A&M University in 2005. A byproduct of this experimentation was the discovery of several new and unusual phenomena, which we will describe through examples. The first is the smallest possible counterexample to the original Shapiro conjecture.

1.3. **The Shapiro conjecture is false for flags in 3-space.** We use σ^b to indicate that the Schubert condition σ is repeated b times and write σ_i for the simple transposition $(i, i{+}1)$. Then $\left(\sigma_2^3,\, \sigma_3^2\right)$ is a Schubert problem for flags of type $\{2,3\}$ in $\mathbb{C}^4$. For distinct points $s,t,u,v,w \in \mathbb{RP}^1$, consider the Schubert intersection

$$X_{\sigma_2}(s) \cap X_{\sigma_2}(t) \cap X_{\sigma_2}(u) \;\cap\; X_{\sigma_3}(v) \cap X_{\sigma_3}(w)\,. \tag{1.5}$$

As flags in projective 3-space, a partial flag of type $\{2,3\}$ is a line ℓ lying on a plane H. Then $(\ell \subset H) \in X_{\sigma_2}(s)$ if ℓ meets the line $\ell(s)$ tangent to γ at $\gamma(s)$, and $(\ell \subset H) \in X_{\sigma_3}(v)$ if H contains the point $\gamma(v)$ on the rational normal curve γ.

Suppose that the flag $\ell \subset H$ lies in the intersection (1.5). Then H contains the two points $\gamma(v)$ and $\gamma(w)$, and hence the secant line $\lambda(v,w)$ that they span. Since ℓ is another line in H, ℓ meets this secant line $\lambda(v,w)$. As $\ell \neq \lambda(v,w)$, it determines H uniquely as the span of ℓ and $\lambda(v,w)$. In this way, we are reduced to determining the lines ℓ which meet the three tangent lines $\ell(s)$, $\ell(t)$, $\ell(u)$, and the secant line $\lambda(v,w)$.

The set of lines which meet the three tangent lines $\ell(s)$, $\ell(t)$, and $\ell(u)$ forms one ruling of a quadric surface Q in $\mathbb{P}^3$. We display a picture of Q and the ruling in Figure 1, as well as the rational normal curve γ with its three tangent lines. This is for a particular choice of s, t, and u, which is described below. The lines meeting $\ell(s)$, $\ell(t)$, $\ell(u)$, and

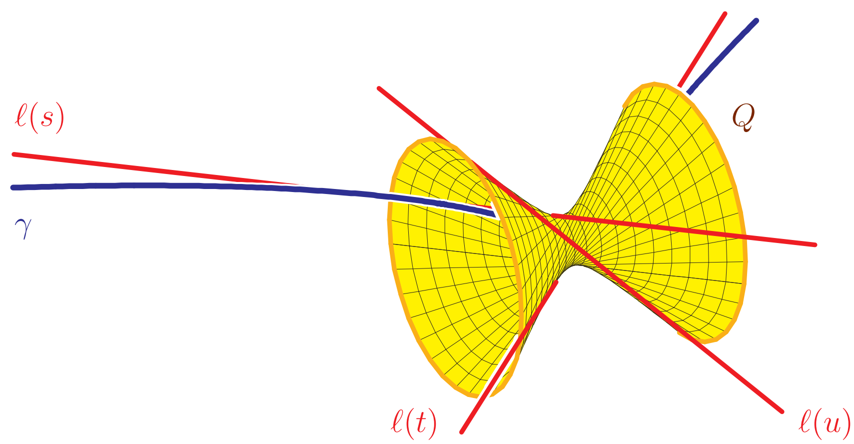


FIGURE 1. Quadric containing three lines tangent to the rational normal curve.

the secant line $\lambda(v,w)$ correspond to the points where $\lambda(v,w)$ meets the quadric Q. In

Figure 2, we display a secant line $\lambda(v, w)$ which meets the hyperboloid in two points, and therefore these choices for v and w give two real flags in the intersection (1.5). There is

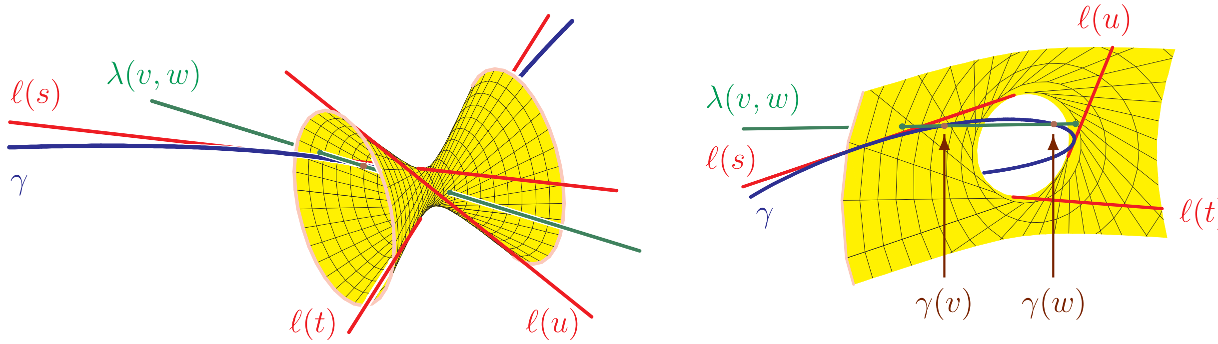


FIGURE 2. Two views of a secant line meeting Q.

also a secant line which meets the hyperboloid in no real points, and hence in two complex conjugate points. For this secant line, both flags in the intersection (1.5) are complex. We show this configuration in Figure 3.

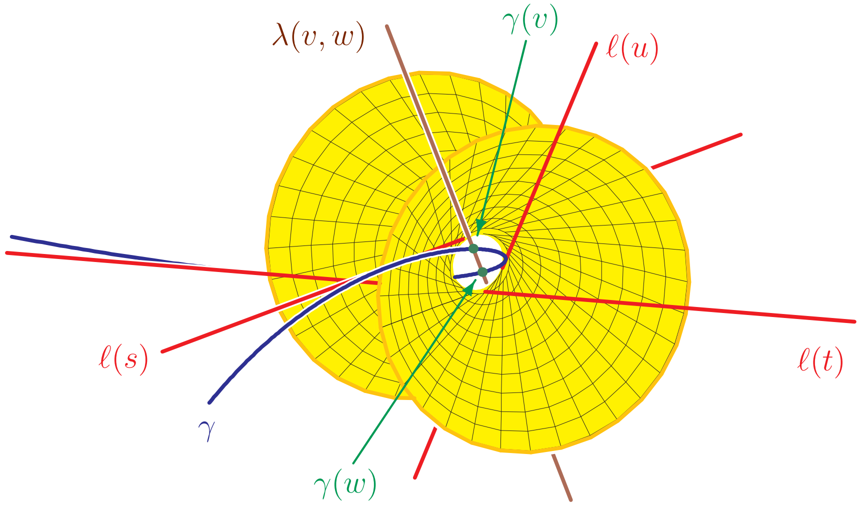


FIGURE 3. A secant line not meeting Q.

To investigate this failure of the Shapiro conjecture, first note that any two parametrizations of two rational normal curves are conjugate under a projective transformation of $\mathbb{P}^3$. Thus it will be no loss to assume that the curve γ has the parametrization

$$\gamma \ : \ t \ \longmapsto \ [2,\ 12t^2-2,\ 7t^3+3t,\ 3t-t^3]\,.$$

Then the lines tangent to γ at the points $(s, t, u) = (-1, 0, 1)$ lie on the hyperboloid

$$x_0^2 - x_1^2 + x_2^2 - x_3^2 \ = \ 0\,.$$

If we parametrize the secant line $\lambda(v, w)$ as $(\frac{1}{2}+l)\gamma(v) + (\frac{1}{2}-l)\gamma(w)$ and then substitute this into the equation for the hyperboloid, we obtain a quadratic polynomial in l, v, w. Its discriminant with respect to l is

$$16(v-w)^2\,(2vw+v+w)(3vw+1)(1-vw)(v+w-2vw)\,. \tag{1.6}$$

We plot its zero-set in the square $v,w\in[-2,2]$, shading the regions where the discriminant is negative. The vertical broken lines are $v,w=\pm1$, the diagonal line is $v=w$, the cross is the value of (v,w) in Figure 2, and the dot is the value in Figure 3. Observe that

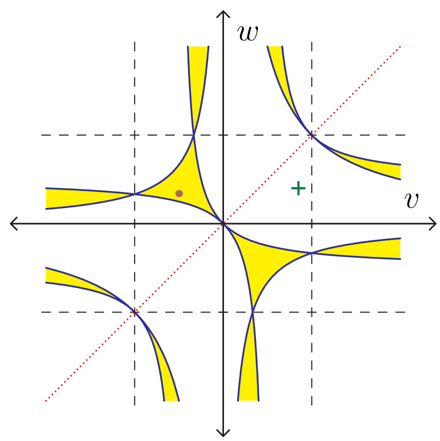


FIGURE 4. Discriminant of the Schubert problem 1.5.

the discriminant is nonnegative if (v,w) lies in one of the squares $(-1,0)^2$, $(0,1)^2$, or if $(\frac{1}{v},\frac{1}{w})\in(-1,1)^2$ and it is positive in the triangles into which the line $v=w$ subdivides these squares. Since $(s,t,u)=(-1,0,1)$, these squares are the values of v and w when both lie entirely within one of the three intervals of $\mathbb{RP}^1$ determined by s,t,u. If we allow Möbius transformations of $\mathbb{RP}^1$, we deduce the following proposition.

Proposition 1.7. *The intersection* (1.5) *is transverse and consists only of real points if there are disjoint intervals I_2 and I_3 of $\mathbb{RP}^1$ so that $s,t,u\in I_2$ and $v,w\in I_3$.*

While this example shows that the Shapiro conjecture is false, Proposition 1.7 suggests that a refinement to the Shapiro conjecture may hold. We will describe such a refinement and present experimental evidence supporting it.

2. RESULTS

Experimentation designed to test hypotheses is a primary means of inquiry in the natural sciences. In mathematics we use proof and example as our primary means of inquiry. Many mathematicians (including the authors) feel that they are striving to understand the nature of objects that inhabit a very real mathematical reality. For us, experimentation plays an important role in helping to formulate reasonable conjectures, which are then studied and perhaps eventually decided.

We first discuss the conjectures which were informed by our experimentation that we describe in Section 4. Then we discuss the proof of these conjectures for the flag manifolds $\mathbb{F}\ell(n-2,n-1;n)$ by Eremenko, Gabrielov, Shapiro, and Vainshtein [5], and an extension of our monotone conjecture which is suggested by their work. Lastly, we present some examples from this experimentation which exhibit new and interesting phenomena.

2.1. **Conjectures.** Let $\alpha = \{\alpha_1 < \cdots < \alpha_k\}$ and n be positive integers with $\alpha_k < n$. Recall that a permutation $w \in W^\alpha$ is Grassmannian if it has a single descent, say at position α_l. Then the Schubert variety $X_w F_\bullet$ of $\mathbb{F}\ell(\alpha; n)$ is the inverse image of the Schubert variety $X_w F_\bullet$ of the Grassmannian $\mathrm{Gr}(\alpha_l, n)$. Write $\delta(w)$ for the unique descent of a Grassmannian permutation w.

A Schubert problem $(w_1, \ldots, w_m)$ for $\mathbb{F}\ell(\alpha; n)$ is *Grassmannian* if each permutation w_i is Grassmannian. A list of points $t_1, \ldots, t_m \in \mathbb{RP}^1$ is *monotone* with respect to a Grassmannian Schubert problem $(w_1, \ldots, w_m)$ if the function

$$t_i \;\longmapsto\; \delta(w_i) \;\in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$$

is monotone, when the ordering of the t_i is consistent with an orientation of $\mathbb{RP}^1$. We also say that the ordered m-tuple $(t_1, \ldots, t_m)$ is a monotone point of $(\mathbb{RP}^1)^m$.

This definition is invariant under the automorphism group of $\mathbb{RP}^1$, which consists of the real Möbius transformations and acts transitively on triples of points on $\mathbb{RP}^1$. Viewing $\mathbb{C}^n$ as the linear space of homogeneous forms on $\mathbb{P}^1$ of degree $n-1$ shows that an automorphism φ of $\mathbb{P}^1$ induces a corresponding automorphism φ of $\mathbb{C}^n$ such that $\varphi(\gamma(t)) = \gamma(\varphi(t))$, and thus $\varphi(F_\bullet(t)) = F_\bullet(\varphi(t))$. The corresponding automorphism φ of $\mathbb{F}\ell(\alpha; n)$ satisfies $\varphi(X_w(t)) = X_w(\varphi(t))$. This was used in the discussion of Section 1.3.

Remark 2.1. Conjecture A of the Introduction involves a monotone choice of points for the Grassmannian Schubert problem $\left(\sigma_2^4,\, \sigma_3^4\right)$ on the flag manifold $\mathbb{F}\ell(2, 3; 5)$. Indeed, $\mathcal{M}_\iota$ is the set of matrices of the form

$$\begin{bmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 0 & 0 & 1 & x_7 & x_8 \end{bmatrix} .$$

The equation $f(s; x) = 0$ is the condition that $E_2(x)$ meets $F_3(s)$ non-trivially, and defines the Schubert variety $X_{\sigma_2}(s)$. Similarly, $g(s; x) = 0$ defines the Schubert variety $X_{\sigma_3}(s)$. The list of points at which f and g were evaluated in Conjecture A is monotone.

Conjecture 2.2. *Suppose that* $(w_1, \ldots, w_m)$ *is a Grassmannian Schubert problem for* $\mathbb{F}\ell(\alpha; n)$. *Then the intersection*

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,, \tag{2.3}$$

is transverse with all points of intersection real, if the points $t_1, \ldots, t_m \in \mathbb{RP}^1$ *are monotone with respect to* $(w_1, \ldots, w_m)$.

We make a weaker conjecture which drops the claim of transversality.

Conjecture 2.4. *Suppose that* $(w_1, \ldots, w_m)$ *is a Grassmannian Schubert problem for* $\mathbb{F}\ell(\alpha; n)$. *Then the intersection* (2.3) *has all points real, if the points* $t_1, \ldots, t_m \in \mathbb{RP}^1$ *are monotone with respect to* $(w_1, \ldots, w_m)$.

Remark 2.5. The example of Section 1.3 illustrates both Conjecture 2.2 and its limitation. The condition on disjoint intervals I_2 and I_3 of Proposition 1.7 is equivalent to the pointss being monotone. The shaded regions in Figure 4, which are the points that give no real solutions, contain no monotone lists of points.

If $\mathbb{F}\ell(\alpha; n)$ is a Grassmannian, then every choice of points is monotone, so Conjecture 2.2 includes the Shapiro conjecture for Grassmannains as a special case. Our experimentation systematically investigated the original Shapiro conjecture for flag manifolds, with a focus on this monotone conjecture. We examined 590 such Grassmannian Schubert problems on 29 different flag manifolds. In all, we verified that each of more than 158 million specific monotone intersections of the form (2.3) had all solutions real. We find this to be overwhelming evidence in support of our monotone conjecture.

Indeed, the set of points $(t_1, \ldots, t_m) \in (\mathbb{P}^1)^m$ where the intersection (2.3) is not transverse is the *discriminant* Σ of the corresponding Schubert problem. This is a hypersurface, unless the intersection is never transverse. The number of real solutions is constant on each connected component of the complement of the discriminant. Conjecture 2.2 asserts that the set of monotone points lies entirely within the region where all solutions are real. Our computations show that the discriminant is a hypersurface for the Grassmannian Schubert problems we considered, and none of the 158 million monotone points we considered was contained in a non-maximal component in which not all solutions were real. While this does not prove Conjecture 2.2 for these problems, it places severe restrictions on the location of the non-maximal components of the complement of the discriminant.

For a given flag manifold, it suffices to know Conjecture 2.4 for *simple Schubert problems*, which involve only simple (codimension 1) Schubert conditions. As simple Schubert conditions are Grassmannian, Conjectures 2.2 and 2.4 apply to simple Schubert problems.

Theorem 2.6. *Suppose that Conjecture* 2.4 *holds for all simple Schubert problems on a given flag manifold* $\mathbb{F}\ell(\alpha; n)$*. Then Conjecture* 2.4 *holds for all Grassmannian Schubert problems on any flag manifold* $\mathbb{F}\ell(\beta; n)$ *where* β *is a subsequence of* α*.*

We prove Theorem 2.6 when $\beta = \alpha$ in Section 3.1 and the general case in Section 3.4.

We give two further and successively stronger conjectures which are supported by our experimental investigation. The first ignores the issue of reality and concentrates only on the transversality of an intersection.

Conjecture 2.7. *If* $(w_1, \ldots, w_m)$ *is a Grassmannian Schubert problem for* $\mathbb{F}\ell(\alpha; n)$ *and the points* $t_1, \ldots, t_m \in \mathbb{RP}^1$ *are monotone with respect to* $(w_1, \ldots, w_m)$*, then the intersection* (2.3) *is transverse.*

Since the set of monotone points is connected, Conjecture 2.7 asserts that it lies in a single component of the complement of the discriminant. Since a main result of [25] is that Conjecture 2.2 holds for simple Schubert problems when the points $t_1, \ldots, t_m$ are sufficiently clustered together, Conjecture 2.7 implies Conjecture 2.2, for simple Schubert problems. Then Theorem 2.6 implies Conjecture 2.4, and the transversality assertion of Conjecture 2.7 implies Conjecture 2.2, without any restriction on the Grassmannian Schubert problem.

Theorem 2.8. *Conjecture* 2.7 *implies Conjecture* 2.2.

Conjecture 2.7 states that for a Grassmannian Schubert problem w, the discriminant Σ contains no points $(t_1,\ldots,t_m)$ that are monotone with respect to w. In our experimentation, we kept track of the non-transverse intersections. None came from monotone points for a Grassmannian Schubert problem. In contrast, there were several hunderd such non-transverse intersections encountered involving non-monotone choices of points. While this does not rule out the existence of monotone choices of points giving a non-transverse intersection, it does suggest that it is highly unlikely.

In every case that we have computed, the discriminant is defined by a polynomial having a special form which shows that Σ contains no points that are monotone with respect to w. We explain this. The set $\Sigma\cap\mathbb{R}^m$ is defined by a single *discriminant polynomial* $\Delta_w(t_1,\ldots,t_m)$, that is well-defined up to multiplication by a scalar. The set of monotone points $(t_1,\ldots,t_m)\in\mathbb{R}^m$ with respect to w has many components. Consider the union of components defined by the inequalities

$$t_i\neq t_j \quad \text{if} \quad i\neq j \qquad \text{and} \qquad t_i<t_j \quad \text{whenever} \quad \delta(w_i)<\delta(w_j)\,. \tag{2.9}$$

For the example of Section 1.3, the region of monotone points is where v,w lie in one of the three intervals of $\mathbb{RP}^1$ defined by s,t,u. As we argued there, we may assume that $(s,t,u)=(-1,0,1)$ and so v,w must lie in one of the three disjoint intervals $(-1,0)$, $(0,1)$, or $(1,-1)$ on $\mathbb{RP}^1$, where the last interval contains ∞. Since any one of these intervals is transformed into any other by a Möbius transformation, it suffices to consider the interval $(0,1)$, which is defined by the inequalities

$$0\ <\ v\,,w\,, \qquad \text{and} \qquad 0\ <\ 1-v\,,\,1-w\,.$$

Note that

$$\begin{aligned} 1-vw &= 1-w+w(1-v)\\ v+w-2vw &= v(1-w)+w(1-v)\,, \end{aligned}$$

which shows that the discriminant (1.6) is positive if $v\neq w$ and $0<v,w<1$.

We conjecture that the discriminant always has such a form for which its positivity (or negativity) on the set (2.9) of monotone points is obvious. More precisely, suppose that $w=(w_1,\ldots,w_m)$ is a Grassmannian Schubert problem for $\mathbb{F}\ell(\alpha;n)$. Set

$$S\ :=\ \{t_i-t_j\mid \delta(w_i)>\delta(w_j)\}\,.$$

Then the set (2.9) of monotone points is

$$\{t=(t_1,\ldots,t_m)\mid g(t)\geq 0 \quad \text{for} \quad g\in S\}\,.$$

Writing $S=\{g_1,\ldots,g_l\}$, the *preorder* generated by S is the set of polynomials of the form

$$\sum_{\varepsilon} c_\varepsilon g_1^{\varepsilon_1}g_2^{\varepsilon_2}\cdots g_l^{\varepsilon_l}\,,$$

where each $\varepsilon_i\in\{0,1\}$ and each coefficient c_ε is a sum of squares of polynomials. Every polynomial in the preorder generated by S is obviously positive on the set (2.9) of monotone points, but not every polynomial that is positive on that set lies in the preorder, at least when $m\geq 5$. Indeed, suppose that $\delta(w_1)\leq\delta(w_2)\leq\cdots\leq\delta(w_m)$. Using the automorphism group of $\mathbb{RP}^1$, we may assume that $t_1=\infty$, $t_2=-1$, $t_3=0$. Then the set (2.9) are those $(t_4,\ldots,t_m)$ such that $0<t_4<\cdots<t_m$. This contains a 2-dimensional cone when $m\geq 5$, so the preorder of polynomials which are positive on this set is not a finitely generated preorder [17, §6.7].

Conjecture 2.10. *Suppose that $(w_1,\ldots,w_m)$ is a Grassmannian Schubert problem for $\mathbb{F}\ell(\alpha;n)$. Then its discriminant Δ_w (or its negative) lies in the preorder generated by the polynomials*

$$S \;:=\; \{t_i-t_j \mid \delta(w_i)>\delta(w_j)\}\,.$$

We showed that this holds for the problem of Section 1.3. Conjecture 2.10 generalizes a conjecture made in [24] that the discriminants for Grassmannians are sums of squares.

Since Conjecture 2.10 implies that the discriminant is nonvanishing on monotone choices of points, it implies Conjecture 2.7, and so by Theorem 2.8, it implies the original Conjecture 2.2. We record this fact.

Theorem 2.11. *Conjecture* 2.10 *implies Conjecture* 2.2.

We give some additional evidence in favor of Conjecture 2.10 in Section 3.5.

2.2. **The result of Eremenko, Gabrielov, Shapiro, and Vainshtein.** Conjecture 2.2 for $\mathbb{F}\ell(n-2,n-1;n)$ follows from a result of Eremenko *et. al* [5]. We discuss this for simple Schubert problems, from which the general case follows, by Theorem 2.6.

There are two types of simple Schubert varieties in $\mathbb{F}\ell(n-2,n-1;n)$,

$$\begin{aligned} X_{\sigma_{n-2}}F_\bullet &:= \{(E_{n-2}\subset E_{n-1}) \mid E_{n-2}\cap F_2\neq\{0\}\}\,, \qquad \text{and}\\ X_{\sigma_{n-1}}F_\bullet &:= \{(E_{n-2}\subset E_{n-1}) \mid E_{n-1}\supset F_1\}\,. \end{aligned}$$

When $n=4$, these are the Schubert varieties $X_{\sigma_2}F_\bullet$ and $X_{\sigma_3}F_\bullet$ of Section 1.3.

Consider the Schubert intersection

$$X_{\sigma_{n-2}}(t_1)\cap\cdots\cap X_{\sigma_{n-2}}(t_p)\;\cap\;X_{\sigma_{n-1}}(s_1)\cap\cdots\cap X_{\sigma_{n-1}}(s_q) \tag{2.12}$$

where $t_1,\ldots,t_p$ and $s_1,\ldots,s_q$ are distinct points in $\mathbb{RP}^1$ and $p+q=2n-1$ with $0<q\leq n$. As in Section 1.3, this Schubert problem is equivalent to one on the Grassmanian $\mathrm{Gr}(n-2,n)$ of codimension 2 planes. The condition that E_{n-1} contains each of the 1-dimensional linear subspaces $\mathrm{span}\{\gamma(s_i)\}$ for $i=1,\ldots,q$ implies that E_{n-1} contains the secant plane $W=\mathrm{span}\{\gamma(s_i)|i=1,\ldots,q\}$ of dimension q. This forces the condition that $\dim W\cap E_{n-2}\geq q-1$, so that $E_\bullet\in X_\tau W$, where τ is the Grassmannian permutation

$$(1,\ 2,\ \ldots,\ n{-}q,\quad n{-}q{+}2,\ \ldots,\ n{-}1,\ \ n{-}q{+}1,\ \ n)\,.$$

One the other hand, when $\dim W\cap E_{n-2}=q-1$, we can recover the hyperplane E_{n-1} by setting $E_{n-1}:=W+E_{n-2}$. Thus the Schubert problem (2.12) reduces to a Schubert problem on $\mathrm{Gr}(n-2,n)$ of the form

$$X_{\sigma_{n-2}}(t_1)\cap\cdots\cap X_{\sigma_{n-2}}(t_p)\;\cap\;X_\tau W\,. \tag{2.13}$$

Using the results of [4], Eremenko, Gabrielov, Shapiro and Vainshtein show that the intersection (2.13) has only real points, when the given points $t_1,\ldots,t_p,s_1,\ldots,s_q$ are monotone with respect to the Schubert problem $(\sigma_{n-2}^p,\,\sigma_{n-1}^q)$.

This suggests a generalization of Conjecture 2.2 to flags of subspaces which are secant to the rational normal curve γ. Let $S := (s_1, s_2, \ldots, s_n)$ be n distinct points in $\mathbb{P}^1$ and for each $i = 1, \ldots, n$, let $F_i(S) := \mathrm{span}\{\gamma(s_1), \ldots, \gamma(s_i)\}$. These subspaces form the flag $F_\bullet(S)$ which is *secant to γ at S*. A list $(S_1, \ldots, S_m)$, of sets of n distinct points in $\mathbb{RP}^1$ is *monotone* with respect to a Grassmannian Schubert problem $(w_1, \ldots, w_m)$ if

(1) There exists a collection of disjoint intervals $I_1, \ldots, I_m$ of $\mathbb{RP}^1$ with $S_i \subset I_i$ for each $i = 1, \ldots, m$, and
(2) If we choose points $t_i \in I_i$ for $i = 1, \ldots, m$, then $(t_1, \ldots, t_m)$ is monotone with respect to the Grassmannian Schubert problem w. This notion does not depend upon the choice of points, as the intervals are disjoint.

Conjecture 2.14. *Given a Grassmannian Schubert problem $(w_1, \ldots, w_m)$ for $\mathbb{F}\ell(\alpha; n)$, the Schubert intersection*

$$X_{w_1}F_\bullet(S_1) \cap X_{w_2}F_\bullet(S_2) \cap \cdots \cap X_{w_m}F_\bullet(S_m)\,,$$

is transverse with all points of intersection real, if the list of subsets $(S_1, \ldots, S_m)$ of $\mathbb{RP}^1$ is monotone with respect to $(w_1, \ldots, w_m)$.

Conjecture 2.14 was formulated in the case when the flag manifolds are Grassmannians in [5], where monotonicity was called well-separatedness. The main result in that paper is its proof for the Grassmannian $\mathrm{Gr}(n{-}2, n)$. A collection $U_1, \ldots, U_r$ of subsets of $\mathbb{RP}^1$ is *well-separated* if there are disjoint intervals $I_1, \ldots, I_r$ of $\mathbb{RP}^1$ with $U_i \subset I_i$ for $i = 1, \ldots, r$.

Proposition 2.15 (Eremenko, *et. al* [5, Theorem 1])**.** *Suppose that $U_1, \ldots, U_r$ is a well-separated collection of finite subsets of $\mathbb{RP}^1$ consisting of $2n-2+r$ points, and with no U_i consisting of a single point. Then there are finitely many codimension 2 planes meeting each of the planes span$\{\gamma(U_i)\}$ for $i = 1, \ldots, r$, and all are real.*

The numerical condition that there are $2n-2+r$ points and that no U_i is a singleton ensures that there will be finitely many codimension 2 planes meeting the subspaces $\mathrm{span}\{\gamma(U_i)\}$. To see how this implies that the intersections (2.13) and (2.12) consist only of real points, let $r = p+1$ and set $U_j := \{t_j, u_j\}$, where the point u_j is close to the point t_j for $j = 1, \ldots, p$ and also set $U_{p+1} := \{s_1, \ldots, s_q\}$. For each $j = 1, \ldots, p$, the limit

$$\lim_{u_j \to t_j} \mathrm{span}\{\gamma(U_j)\}$$

is the 2-plane osculating the rational normal curve at t_j. The condition that the subsets $U_1, \ldots, U_{p+1}$ are are well-separated implies that the points $\{s_1, \ldots, s_q, t_1, \ldots, t_p\}$ are monotone with respect to the Schubert problem $(\sigma_{n-2}^p, \sigma_{n-1}^q)$. Thus the intersection (2.13) is a limit of intersections of the form in Proposition 2.15, and hence consists only of real points. This gives the following corollary to Proposition 2.15, also proven in [5].

Corollary 2.16. *Suppose that there exist disjoint intervals $I \supset \{s_1, \ldots, s_q\}$ and $J \supset \{t_1, \ldots, t_p\}$. Then all codimension 2 planes in the intersection* (2.12) *are real. Thus all flags $E_\bullet \in \mathbb{F}\ell(n{-}2, n{-}1; n)$ in the intersection* (2.13) *are real.*

We have not yet investigated Conjecture 2.14, and the results of [5] are the only evidence currently in its favor. We believe that experimentation testing this conjecture, in the spirit of the experimentation described in Section 4, is a natural and worthwhile next step.

2.3. **Examples.** While the original goal of our experimentation was to study Conjecture 2.2, this project became a general study of Schubert intersection problems on small flag manifolds. Here, we report on some new and interesting phenomena which we observed, beyond support for Conjecture 2.2.

We first discuss some of the Schubert problems that we investigated, presenting in tabular form the data from our experimentation on those problems. Some of these appear to present new or interesting phenomena beyond Conjecture 2.2. We next discuss some phenomena that we observed in our data, and which we can establish rigorously. One is the smallest enumerative problem that we know of with an unexpectedly small Galois group [9, 28], and the other is a Schubert problem for which the intersection is not transverse, when the given flags osculate the rational normal curve.

A Schubert intersection of the form

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)$$

may be encoded by labeling each point $t_i \in \mathbb{RP}^1$ with the corresponding Schubert condition w_i. The automorphism group of $\mathbb{RP}^1$ acts on the flag variety $\mathbb{F}\ell(\alpha; n)$, and hence on collections of labeled points. A coarser equivalence which captures the combinatorics of the arrangement of Schubert conditions along $\mathbb{RP}^1$ is isotopy, and isotopy classes of such labeled points are called *necklaces*, which are the different arrangements of m beads labeled with $w_1, \ldots, w_m$ and strung on the circle $\mathbb{RP}^1$. Our experimentation was designed to study how the number of real solutions to a Schubert problem was affected by the necklace. *Monotone necklaces* are necklaces corresponding to monotone choices of points.

To that end, we kept track of the number of real solutions to a Schubert problem by the associated necklace, and have archived the results in linked web pages available at `www.math.tamu.edu/~sottile/pages/Flags/`. Section 4 discusses how these computations were carried out. While Conjecture 2.2 is the most basic assertion that we believe is true, there were many other phenomena, both general and specific, that our experimentation uncovered. We describe some of them below. Conjecture 3.8 and Theorem 3.13 are some others. Our data contain many more interesting examples, and invite the interested reader browse the data online.

2.3.1. *Conjecture* 2.2. Table 1 shows the data from computing 3.2 million instances of the Schubert problem $({\sigma_2}^4, {\sigma_3}^4)$ on $\mathbb{F}\ell(2,3;5)$ underlying Conjecture A from the Introduction. Each row corresponds to a necklace, and the entries record how often a given number of real solutions was observed for the corresponding necklace. Representing the Schubert conditions σ_2 and σ_3 by their subscripts, we may write each necklace linearly as a sequence of 2s and 3s. The only monotone necklace is in the first row, and Conjecture 2.2 predicts that any intersection with this necklace will have all 12 solutions real, as we observe.

The other rows in this table are equally striking. It appears that there is a unique necklace for which it is possible that no solutions are real, and for five of the necklaces, the minimum number of real solutions is 4. The rows in this and all other tables are ordered to highlight this feature. Every row has a non-zero entry in its last column. This implies that for every necklace, there is a choice of points on $\mathbb{RP}^1$ with that necklace for

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
22223333	0	0	0	0	0	0	400000
22322333	0	0	118	65425	132241	117504	84712
22233233	0	0	104	65461	134417	117535	82483
22332233	0	0	1618	57236	188393	92580	60173
22323323	0	0	25398	90784	143394	107108	33316
22332323	0	2085	79317	111448	121589	60333	25228
22232333	0	7818	34389	58098	101334	81724	116637
23232323	15923	41929	131054	86894	81823	30578	11799

TABLE 1. The Schubert problem $({\sigma_2}^4, {\sigma_3}^4)$ on $\mathbb{F}\ell(2,3;5)$.

which all 12 solutions are real. Since this is a simple Schubert problem, that feature is a consequence of Corollary 2.2 of [23].

Table 2 shows data from a related problem $({\sigma_1}^2, {\sigma_2}^3, {\sigma_3}^3, {\sigma_4}^2)$ with 12 solutions. We only computed three necklaces for this problem, as it has 1,272 necklaces. In the necklaces,

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
1122233344	0	0	0	0	0	0	10000
1122244333	0	0	0	0	0	0	10000
1133322244	0	102	462	1556	3821	2809	1250

TABLE 2. The Schubert problem $({\sigma_1}^2, {\sigma_2}^3, {\sigma_3}^3, {\sigma_4}^2)$ on $\mathbb{F}\ell(1,2,3,4;5)$.

i represents the Schubert condition σ_i. The only monotone necklace is in the first row. While the second row is not monotone, it appears to have only real solutions. A similar phenomenon (some non-monotone necklaces having only real solutions) was observed in other Schubert problems involving 4- and 5-step flag manifolds. This can be seen in the example of Table 3, as well as the third part of Theorem 3.19.

Table 3 shows data from the problem $({\sigma_1}^2, {\sigma_2}^2, 246, \sigma_3, {\sigma_4}^2, {\sigma_5}^2)$ on $\mathbb{F}\ell(1,2,3,4,5;6)$ with 8 solutions. In the necklaces, i represents σ_i and C represents the Grassmannian condition 246 with descent at 3. We only computed 13 necklaces for this problem, as it has 11,352 necklaces. Note that three non-monotone necklaces have only real solutions, one has at least 6 solutions, and 7 have at least 4 real solutions.

2.3.2. *Apparent lower bounds.* In the last section, we noted that the lower bound on the number of real solutions seems to depend upon the necklace. We also found many Schubert problems with an apparent *lower bound* which holds for *all* necklaces. For example, Table 4 is for the Schubert problem $(\sigma_3, (1362)^2, {\sigma_4}^2, 1346)$ on $\mathbb{F}\ell(3,4;7)$, which has degree 10. We only display 4 of the 16 necklaces for this problem. Here a, b, c, d

Necklace	Number of Real Solutions				
	0	2	4	6	8
1122C34455	0	0	0	0	50000
11C3445522	0	0	0	0	50000
1122C35544	0	0	0	0	50000
11C3554422	0	0	0	0	50000
115522C344	0	0	0	3406	46594
11C3552244	0	0	5401	24714	19885
1155C34422	0	0	6347	19567	24086
112255C344	0	0	7732	23461	18807
11C3442255	0	0	12437	20396	17167
114422C355	0	0	12508	19177	18315
11445522C3	0	0	15109	25418	9473
11554422C3	0	0	17152	23734	9114
135241C524	298	7095	18280	17871	6456

TABLE 3. The Schubert problem $({\sigma_1}^2, {\sigma_2}^2, 246, \sigma_3, {\sigma_4}^2, {\sigma_5}^2)$ on $\mathbb{F}\ell(1,2,3,4,5;6)$.

Necklace	Number of Real Solutions				
	0	2	4	6	8	10
abbccd	0	0	0	0	0	100000
acbbcd	0	0	0	16722	50766	32512
accbbd	0	0	11979	26316	29683	32022
acbdbc	0	0	27976	34559	26469	10996

TABLE 4. The Schubert problem $(\sigma_3, (1362)^2, {\sigma_4}^2, 1346)$ on $\mathbb{F}\ell(3,4;7)$.

refer to the four conditions $(\sigma_3, 1362, \sigma_4, 1346)$. There are four other necklaces giving a monotone choice of points, and for those the solutions were always real. None of the remaining 8 necklaces had fewer than four real solutions.

Such lower bounds on the number of real solutions to enumerative geometric problems were first found by Eremenko and Gabrielov [3] in the context of the Shapiro conjecture for Grassmannians. Lower bounds have also been proven for problems of enumerating rational curves on surfaces [10, 13, 30] and for some sparse polynomial systems [19]. We do not yet know a reason for the lower bounds here.

2.3.3. *Apparent upper bounds.* On $\mathbb{F}\ell(1,2,3,4;5)$, set $A := 1325$ and $B := 2143$. The Schubert problem (A^2, B^3) has degree 7, but none of the 1 million instances we computed had more than 5 real solutions.

Neither condition A nor B is Grassmannian, and so this Schubert problem is not related to the conjectures in this paper.

Necklace	Number of Real Solutions			
	1	3	5	7
$AABBB$	0	500000	0	0
$ABABB$	193849	268969	37182	0

TABLE 5. The Schubert problem (A^2, B^3) on $\mathbb{F}\ell(1,2,3,4;5)$.

2.3.4. *Apparent gaps.* On $\mathbb{F}\ell(1,3,5;6)$, set $A := 21436$ and $B := 31526$. The Schubert problem $(A^2, B, {\sigma_3}^2)$ has degree 8 and it appears to exhibit gaps in the possible numbers of real solutions. Table 6 gives the data from this computation. In each necklace, 3 represents the Grassmannian condition σ_3. This is a new phenomena first observed in

Necklace	Number of Real Solutions				
	0	2	4	6	8
AAB33	0	0	991894	0	8106
AA3B3	111808	0	888040	0	152
A3A3B	311285	0	681416	0	7299
A33AB	884186	0	115814	0	0

TABLE 6. The Schubert problem $(A^2, B, {\sigma_3}^2)$ on $\mathbb{F}\ell(1,3,5;6)$.

some sparse polynomial systems [19, § 7].

2.3.5. *Small Galois group.* One unusual problem that we looked at was on the flag manifold $\mathbb{F}\ell(2,4;6)$ and it involved four identical non-Grassmannian conditions, 1425. We can prove that this problem has six solutions, and that they are always all real.

Theorem 2.17. *For any distinct $s,t,u,v \in \mathbb{RP}^1$, then intersection*

$$X_{1425}(s) \cap X_{1425}(t) \cap X_{1425}(u) \cap X_{1425}(v)$$

is transverse and consists of 6 *real points.*

This Schubert problem exhibits some other exceptional geometry concerning its Galois group, which we now define. Let $(w_1, \ldots, w_s)$ be a Schubert problem for $\mathbb{F}\ell(\alpha;n)$ and consider the configuration space of s-tuples of flags $(F^1_\bullet, F^2_\bullet, \ldots, F^s_\bullet)$ for which

$$X \;:=\; X_{w_1}F^1_\bullet \cap X_{w_2}F^2_\bullet \cap \cdots \cap X_{w_s}F^s_\bullet$$

is transverse and hence X consists of finitely many points. If we pick a basepoint of this configuration space and follow the intersection along a based loop in the configuration space, we will obtain a permutation of the intersection X corresponding to the base point. Such permutations generate the *Galois group* of this Schubert problem.

Harris [9] defined Galois groups for any enumerative geometric problem and Vakil [28] investigated them for Schubert problems on Grassmannians, showing that many problems have a Galois group that contains at least the alternating group. He also found some Schubert problems on Grassmannian whose Galois group is not the full symmetric group. This Schubert problem also has a strikingly small Galois group, and is the simplest Schubert problem we know with a small Galois group.

Theorem 2.18. *The Galois group of the Schubert problem* $(1425)^4$ *on* $\mathbb{F}\ell(2,4;6)$ *is the symmetric group on* 3 *letters.*

We prove both theorems. First, consider the Schubert variety $X_{1425}F_\bullet$

$$X_{1425}F_\bullet \;=\; \{E_2 \subset E_4 \mid \dim E_2 \cap F_3 \geq 1 \text{ and } \dim E_4 \cap F_3 \geq 2\}.$$

The image of $X_{1425}F_\bullet$ under the projection $\pi_4 \colon \mathbb{F}\ell(2,4;6) \twoheadrightarrow \mathrm{Gr}(4,6)$ is

$$\Omega_{1245}F_\bullet \;:=\; \{E_4 \in \mathrm{Gr}(4,6) \mid \dim E_4 \cap F_3 \geq 2\}.$$

Since this Schubert variety has codimension 2 in $\mathrm{Gr}(4,6)$, a variety of dimension 8, there are finitely many 4-planes E_4 which have Schubert position 1245 with respect to four general flags. In fact, there are exactly 3. (See Section 8.1 of [22], which treats the dual problem in $\mathrm{Gr}(2,6)$.)

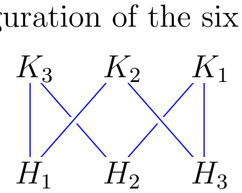
Thus we have a fibration of Schubert problems

$$\bigcap_{i=1}^{4} X_{1425}F^i_\bullet \xrightarrow{\;\pi_4\;} \bigcap_{i=1}^{4} \Omega_{1245}F^i_\bullet\,. \tag{2.19}$$

Let K be a solution to the Schubert problem in $\mathrm{Gr}(4,6)$. We ask, for which 2-planes H in $\mathbb{C}^6$ is the flag $H \subset K$ a solution to the Schubert problem in $\mathbb{F}\ell(2,4;6)$? From the description of $X_{1425}F_\bullet$, H must be a 2-plane in K which meets each linear subspace $K \cap F^i_3$ non-trivially. As K lies in each Schubert cell $\Omega^\circ F^i_\bullet$, $K \cap F^i_3$ is a 2-plane. Thus we are looking for the 2-planes H in K which meet four general 2-planes $K \cap F^i_3$. There are two such 2-planes H, as this is an instance of the problem of lines in $\mathbb{P}^3$ meeting four lines. We conclude that there are six solutions to the Schubert problem on $\mathbb{F}\ell(2,4;6)$.

This Schubert problem projects to one in $\mathrm{Gr}(2,6)$ with three solutions that is dual to the projection in $\mathrm{Gr}(4,6)$. Let H_i and K_i for $i = 1,2,3$ be the 2-planes and 4-planes which are solutions to the two projected problems. For each K_i there are exactly two H_j for which $H_j \subset K_i$ is a solution to the original problem in $\mathbb{F}\ell(2,4;6)$. Dually, for each H_i there are exactly two K_j for which $H_i \subset K_j$ is a solution to the original problem. There is only one possibility for the configuration of the six flags, up to relabeling:

(2.20)



Proof of Theorem 2.17. Since the flags osculate the rational normal curve, the problems obtained by projecting the intersection in Theorem 2.17 to Grassmannians have only real solutions, as shown in Theorem 3.9 of [24]. Thus all subspaces H_i and K_i in (2.20) are real, and so the six solution flags of (2.20) are all real. □

Proof of Theorem 2.18. Since the six solution flags have the configuration given in (2.20), we see that any permutation of the six solutions is determined by its action on the three

4-planes K_1, K_2, K_3. Thus the Galois group is at most the symmetric group S_3. The explicit description given in Section 8.1 of [22] and also the analysis of Vakil [28] shows that the Galois group of the projected problem in $\mathrm{Gr}(4,6)$ is S_3. $\square$

2.3.6. *A non-transverse Schubert problem.* Our experimentation uncovered a Schubert problem whose corresponding intersection is not transverse or even proper, when it involves flags osculating a rational normal curve. This may have negative repercussions for part of Varchenko's program on the Bethe Ansatz and Fuchsian equations [14]. This was unexpected, as Eisenbud and Harris showed that on a Grassmannian, any intersection

$$X_{w_1}(t_1) \cap \cdots \cap X_{w_m}(t_m) \tag{2.21}$$

is proper in that it has the expected dimension $\dim(\alpha) - \sum \ell(w_i)$, if the points $t_1, \ldots, t_m$ in $\mathbb{P}^1$ are distinct [1, Theorem 2.3]. On any flag manifold, if each condition (except possibly one) has codimension 1 ($\ell(w_i) = 1$), and if the points $t_1, \ldots, t_m \in \mathbb{P}^1$ are general, then the intersection (2.21) is transverse, and hence proper [23, Theorem 2.1]. We show this is not the case for all Schubert problems on the flag manifold.

The manifold of flags of type $\{1,3\}$ in $\mathbb{C}^5$ has dimension 8. Since $\ell(32514) = 5$ and $\ell(21435) = 2$, there are no flags of type $\{1,3\}$ satisfying the Schubert conditions $(325,\ (214)^2)$ imposed by three general flags. This is not the case if the flags osculate a rational normal curve γ.

Theorem 2.22. *The intersection $X_{325}(u) \cap X_{214}(s) \cap X_{214}(t)$ is nonempty for all $s, t, u \in \mathbb{P}^1$.*

Proof. We may assume without any loss that $u = \infty$, so that flags in $X^\circ_{325}(u)$ are given by matrices in $\mathcal{M}_{325}$. Consider the 3×5 matrix in $\mathcal{M}_{325}$.

$$\begin{bmatrix} 0 & 0 & 1 & \frac{3}{2}(s+t) & 6st \\ 0 & 1 & 0 & -3st & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.23}$$

Let $E_\bullet \colon E_1 \subset E_3$ be the corresponding flag. We will show that $E_\bullet \in X_{214}(s) \cap X_{214}(t)$. Let v_1, v_2, and v_3 to be the row vectors in (2.23). Consider the dual vector

$$\lambda(s) \ := \ (s^4,\ -4s^3,\ 6s^2,\ -4s,\ 1)\,,$$

and note that $\lambda(s)$ annihilates $\gamma(s)$, $\gamma'(s)$, $\gamma''(s)$, and $\gamma'''(s)$, so that $\lambda(s)$ is a linear form annihilating the 4-plane $F_4(s)$ osculating the rational normal curve γ at the point $\gamma(s)$. Note that $v_1 \cdot \lambda(s)^t = 0$, so that $E_1 \subset F_4(s)$. Also,

$$\gamma'(s) \ = \ v_2 + 2sv_1 + (4s^3 - 12s^2t)v_3\,,$$

and so $E_3 \cap F_2(s) \neq 0$. In particular this implies that $E_\bullet \in X_{214}(s)$. We similarly have that $E_\bullet \in X_{214}(t)$. $\square$

3. Discussion

We establish relationships between the different conjectures of Section 2, between the conjectures for different Schubert problems on the same flag manifold, and between the conjectures for Schubert problems on different flag manifolds. This includes a proof of Theorem 2.6 and a subtle generalization of Conjecture 2.2. We conclude by proving Conjecture 2.10 for several Schubert problems.

3.1. **Child problems.** The Bruhat order on W^α is defined by its covers $w \lessdot u$: if $\ell(w)+1=\ell(u)$ and $w^{-1}u$ is a transposition (b,c). Necessarily, there exists an i such that $b \leq \alpha_i < c$, but this number i may not be unique. Write $w \lessdot_i u$ when $w \lessdot u$ in the Bruhat order and the transposition $(b,c) := w^{-1}u$ satisfies $b \leq \alpha_i < c$. This defines the cover relation in a partial order $<_i$ on W^α, which is a subposet of the Bruhat order, and is called the α_i-Bruhat order in the combinatorics literature [20]. When $w < u$ are two Grasmannian permutations with the same descent α_i which are related in Bruhat order, then $w <_i u$ and there is a chain of covers in the $<_i$-order connecting w to u.

Suppose that $(v, w_1, w_2, \ldots, w_m)$ is a Schubert problem for $\mathbb{F}\ell(\alpha; n)$ and that $v = \sigma_{\alpha_i}$. For any permutation u with $w_1 \lessdot_i u$, we have $\ell(v) + \ell(w_1) = \ell(u)$ and so $(u, w_2, \ldots, w_m)$ is a Schubert problem for $\mathbb{F}\ell(\alpha; n)$. We say that $(u, w_2, \ldots, w_m)$ is a *child problem* of the original Schubert problem $(v, w_1, w_2, \ldots, w_m)$ and write

$$(v, w_1, w_2, \ldots, w_m) \prec\!\!\cdot\; (u, w_2, \ldots, w_m)\,,$$

which defines the covering relation for a partial order $\prec$ on the set of Schubert problems for $\mathbb{F}\ell(\alpha; n)$. Since every cover $w \lessdot u$ in the Bruhat order on W^α has the form $\lessdot_i$ for some i, the minimal elements in this partial order $\prec$ are exactly the simple Schubert problems. The reason for these definitions is the following lemma.

Lemma 3.1. *Suppose that $(v, w_1, w_2, \ldots, w_m) \prec\!\!\cdot (u, w_2, \ldots, w_m)$ is a cover between two Grassmannian Schubert problems for $\mathbb{F}\ell(\alpha; n)$, where $\delta(w_1) = \alpha_i$, $v = \sigma_{\alpha_i}$, and $w_1 \lessdot_i u$. If Conjecture 2.4 holds for $(v, w_1, w_2, \ldots, w_m)$, then it holds for $(u, w_2, \ldots, w_m)$.*

The case $\beta = \alpha$ of Theorem 2.6 follows from Lemma 3.1 as any Grassmannian Schubert problem is connected to a simple Schubert problem via a chain of covers as in Lemma 3.1. In turn, Lemma 3.1 is a consequence of Lemma 3.3, which is proven in the next section.

3.2. **Limits of Schubert intersections.** Let $w \in W^\alpha$ be a Schubert condition for $\mathbb{F}\ell(\alpha; n)$ and suppose that $v = \sigma_{\alpha_i}$. If $t \neq 0$, then the intersection $X_w(0) \cap X_v(t)$ is (generically) transverse. One result of [25] concerns the limit of this intersection. Specifically, we have the cycle-theoretic equality

$$\lim_{t\to 0} X_w(0) \cap X_v(t) \;=\; \sum_{w \lessdot_i u} X_u(0)\,. \tag{3.2}$$

That is, the support of the scheme-theoretic limit is the union of Schubert varieties in the sum, and this scheme-theoretic limit is reduced at the generic point of each Schubert variety in the sum. We use this to prove the following lemma.

Lemma 3.3. *Let $(v, w_1, w_2, \ldots, w_m)$ be a Schubert problem for $\mathbb{F}\ell(\alpha; n)$, where $v = \sigma_{\alpha_i}$. Suppose that $t_2, \ldots, t_m$ are negative real numbers such that the intersection*

$$X_v(t) \cap X_{w_1}(0) \,\cap\, X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)$$

consists only of real points, for any positive number t. Then, for any permutation u with $w_1 \lessdot_i u$, the intersection

$$X_u(0) \,\cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)$$

consists only of real points

Proof. Set $Y := X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)$. We assumed that if $0 < t$, then $X_{w_1}(0) \cap X_v(t) \cap Y$ consists only of real points. The property of only having real points of intersection is preserved under taking limits, and so (3.2) implies that every point of

$$Y \,\cap \sum_{w_1 \lessdot_i u} X_u(0)$$

is real. In particular, if $w_1 \lessdot_i u$, then $Y \cap X_u(0)$ consists only of real points. $\square$

Proof of Lemma 3.1. Let $t_1, \ldots, t_m \in \mathbb{RP}^1$ be a monotone choice of points for the Schubert problem $(u, w_2, \ldots, w_m)$. Applying a real Möbius transformation if necessary, we may assume that $t_1 = 0$ and that $t_2, \ldots, t_m$ are negative real numbers. Thus it suffices to show that

$$X_u(0) \,\cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m) \tag{3.4}$$

consists only of real points. Since $\delta(u) = \delta(w_1) = \delta(v) = \alpha_i$, it follows that if $0 < t$, then $(t, 0, t_2, \ldots, t_m)$ is monotone with respect to the Schubert problem $(v, w_1, w_2, \ldots, w_m)$. By our assumption that Conjecture 2.4 holds for $(v, w_1, w_2, \ldots, w_m)$, the intersection

$$X_v(t) \cap X_{w_1}(0) \,\cap\, X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)$$

consists only of real points, for any positive number t. But then Lemma 3.3 implies that the intersection (3.4) consists only of real points. $\square$

3.3. **Refined monotone conjecture.** Lemma 3.3 leads to an extension of Conjecture 2.2 to some cases when the Schubert problem is not Grassmannian. We first give an example, which indicates a strengthening of Theorem 2.6.

Example 3.5. Consider the following instance of the cycle-theoretic equality (3.2),

$$\lim_{x \to 0^+} X_{142}(0) \cap X_{\sigma_3}(x) \;=\; X_{152}(0) \cup X_{143}(0)\,. \tag{3.6}$$

Note that $\delta(142) = 2$. Suppose that Conjecture 2.2 holds for the Schubert problem $(\sigma_2^3, 142, \sigma_3^3)$. Then, if $s < t < u < 0 < x < y < z$, the intersection

$$X_{\sigma_2}(s) \cap X_{\sigma_2}(t) \cap X_{\sigma_2}(u) \,\cap\, X_{142}(0) \cap X_{\sigma_3}(x) \,\cap X_{\sigma_3}(y) \cap X_{\sigma_3}(z)$$

consists only of real points, as the choice of points $s, t, u, 0, x, y, z$ is monotone with respect to the given Schubert problem. As in the proof of Lemma 3.3, the limit (3.6) implies that whenever $s < t < u < 0 < y < z$, the intersection

$$X_{\sigma_2}(s) \cap X_{\sigma_2}(t) \cap X_{\sigma_2}(u) \,\cap\, X_{143}(0) \,\cap X_{\sigma_3}(y) \cap X_{\sigma_3}(z)$$

consists only of real points, even though the permutation 14325 is not Grassmannian.

We extend our notion of monotone choices of points to encompass this last example. For a permutation $w \in W^\alpha$, let $\delta(w) \subset \{\alpha_1, \dots, \alpha_k\}$ be its set of descents. Given two subsets $S, T \subset \{\alpha_1, \dots, \alpha_k\}$, we say that S *preceeds* T, written $S < T$ if we have $i \leq j$ for all $i \in S$ and $j \in T$. This does not define a partial order on the set of subsets, but it does give a notion of when a list of subsets is increasing. For example

$$\{2\} \ < \ \{2\} \ < \ \{2\} \ < \ \{2,3\} \ < \ \{3\} \ < \ \{3\} \tag{3.7}$$

is increasing, but $\{2,3\} \not< \{2,3\}$. Note that $\{2\} < \{2\}$.

A list of points $(t_1, \dots, t_m) \in \mathbb{RP}^1$ is monotone with respect to a Schubert problem $(w_1, \dots, w_m)$ for $\mathbb{F}\ell(\alpha; n)$ if the function

$$t_i \ \longmapsto \ \delta(w_i) \subset \{\alpha_1, \dots, \alpha_k\}$$

is monotone, when the ordering of the t_i is consistent with some ordering of $\mathbb{RP}^1$. For example, $(s < t < u < 0 < y < z)$ is monotone with respect to the Schubert problem $(\sigma_2, \sigma_2, \sigma_2, 143, \sigma_3, \sigma_3)$, as $\delta(143) = \{2,3\}$, and we have (3.7). We give a refinement of Conjecture 2.2, which drops the condition that the Schubert problem is Grassmannian.

Conjecture 3.8. *Suppose that $(w_1, \dots, w_m)$ is a Schubert problem for $\mathbb{F}\ell(\alpha; n)$. Then the intersection*

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,, \tag{3.9}$$

is transverse with all points of intersection real, if the points $t_1, \dots, t_m \in \mathbb{RP}^1$ are monotone with respect to $(w_1, \dots, w_m)$.

Remark 3.10. There are many Schubert problems for which there are no monotone points. For example, two of the conditions (A) in the Schubert problem of Table 5 have descent set $\{2,4\}$ and so there are no monotone points. As reported there, for each of the two different necklaces, there are choices of points with not all solutions real. Similarly, in the Schubert problem of Table 6, there are three permutations with descent set $\{1,3,5\}$, and thus no monotone points. The Schubert problem in Section 2.3.5 consists of four identical conditions w with $\delta(w) = \{2,4\}$, and so there are no monotone points. Nevertheless, we showed that all solutions are real.

The other conjectures of Section 2.1 may be refined to include this more general notion of monotone points. For example, we conjecture that the discriminant of a Schubert problem does not vanish for monotone points, and that it (or its negative) lies in the preorder generated by differences of the t_i, as in Conjecture 2.10.

The theorems of Section 2.1 also hold in this generality, as the proofs are identical. For example, we have the following strengthening of Theorem 2.6.

Theorem 2.6′. *Suppose that Conjecture* 3.8 *holds for all simple Schubert problems on a given flag manifold, $\mathbb{F}\ell(\alpha; n)$. Then Conjecture* 3.8 *holds for all Schubert problems on any flag manifold $\mathbb{F}\ell(\beta, n)$ where β is any subsequence of α. (Here, the condition of transversality in Conjecture* 3.8 *is dropped.)*

Example 3.11. Table 7 shows data from the Schubert problem $(\sigma_2{}^2, 1432, 135\,2, 1254, \sigma_4{}^2)$ on $\mathbb{F}\ell(2,3,4;6)$, which has 12 solutions, and involves two non-Grassmannian conditions. In the necklaces, $2, A, 3, B, 4$ represent the five Schubert conditions, respectively. Their descent sets are $\{2\}, \{2,3\}, \{3\}, \{3,4\}, \{4\}$, so only the first row is monotone, and these

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
22A3B44	0	0	0	0	0	0	7500
22AB443	0	0	0	0	0	0	7500
22AB344	0	0	0	0	306	3776	3416
22B3A44	0	0	0	12	1359	3446	2683
22344AB	0	0	0	1213	2129	1771	2387

TABLE 7. The Schubert problem $(\sigma_2{}^2, 1432, 1354, 1254, \sigma_4{}^2)$ on $\mathbb{F}\ell(2,3,4;6)$.

data support Conjecture 3.8. We only show 5 of the 90 necklaces.

3.4. **Projections.** Suppose that β is a subsequence of α. In Section 1.1 we considered projections $\pi\colon \mathbb{F}\ell(\alpha;n) \to \mathbb{F}\ell(\beta;n)$ obtained by forgetting the components of a flag $E_\bullet \in \mathbb{F}\ell(\alpha;n)$ with dimension in $\alpha \setminus \beta$. The image $\pi(X_w F_\bullet)$ of a Schubert variety of $\mathbb{F}\ell(\alpha;n)$ is a Schubert variety of $\mathbb{F}\ell(\beta;n)$ for a (possibly) different permutation $\pi(w)$. Recall that $w \in W^\alpha$ is a permutation whose descents can only occur at positions in α. The permutation $\pi(w)$ is obtained by ordering the values of w between successive positions in β. For example, if $n = 9$, $\alpha = \{2,4,5,7\}$ and $\beta = \{2,7\}$, then

$$\pi(13\,\underline{58\,4\,27}\,69) \;=\; 13\,\underline{24578}\,69 \qquad \text{and} \qquad \pi(26\,\underline{45\,7\,19}\,36) \;=\; 26\,\underline{14579}\,36\,.$$

Because $\pi(X_w F_\bullet(s)) = X_{\pi(w)} F_\bullet(s)$, if we have a Schubert problem $(w_1, \ldots, w_m)$ on $\mathbb{F}\ell(\alpha;n)$ and m general flags, then π is a map between the intersections

$$\pi \;:\; X_{w_1}(t_1) \cap \cdots \cap X_{w_m}(t_m) \;\longrightarrow\; X_{\pi(w_1)}(t_1) \cap \cdots \cap X_{\pi(w_m)}(t_m)\,. \tag{3.12}$$

Suppose that both $(w_1, \ldots, w_m)$ and $(\pi(w_1), \ldots, \pi(w_m))$ are Schubert problems. Then the map π of (3.12) is a fibration with finite fibres. If the two problems have the same same degree, then π is an isomorphism. In that case, we say that $(\pi(w_1), \ldots, \pi(w_m))$ is a *projection* of $(w_1, \ldots, w_m)$ and that $(w_1, \ldots, w_m)$ is a *lift of* $(\pi(w_1), \ldots, \pi(w_m))$.

Theorem 3.13. *Suppose that the Schubert problem $w := (w_1, \ldots, w_m)$ on $\mathbb{F}\ell(\alpha;n)$ is a lift of the Schubert problem $\pi(w) = (\pi(w_1), \ldots, \pi(w_m))$ on $\mathbb{F}\ell(\beta;n)$. If Conjecture 3.8 holds for $\pi(w)$ then it holds for w.*

Proof. Suppose that the permutations in w are ordered so that

$$\delta(w_1) \;<\; \delta(w_2) \;<\; \cdots \;<\; \delta(w_m)$$

and let $t_1 < \cdots < t_m$ be real numbers. Then $\delta(\pi(w_1)) < \cdots < \delta(\pi(w_m))$ and our assumption on $\pi(w)$ implies that the right-hand intersection in (3.12) consists only of real points. Since the map π in (3.12) is an isomorphism, we conclude that the left-hand intersection in (3.12) consists only of real points. $\square$

Example 3.14. Projection and lifts relate Schubert problems in many ways. The Grassmannian Schubert problem $w := (4\,1235, 15\,234, 135\,24, 1345\,2, 12456\,)$ on $\mathbb{F}\ell(1,2,3,4,5;6)$ has degree 4 and and it projects to the Schubert problem $(\sigma_3, 125, 135, 134, \sigma_3)$ on the Grassmannian $G(3,6)$, which also has degree 4. One may compute a discriminant (as in [24, §3E]) to show that the Shapiro conjecture holds for this Schubert problem. But then every Shapiro-type intersection for w has all solutions real, and thus Conjecture 2.2 holds for w. More interestingly, the projection of w to $\mathbb{F}\ell(2,4;6)$ also has only real solutions. This is the problem $(14\,23, 15\,23, 1325, 1345, 1245)$ of degree 4. Since the conditions have descents $(\{2\}, \{2\}, \{2,4\}, \{4\}, \{4\})$, there is a monotone choice of points, and so Conjecture 3.8 holds for this last Schubert problem.

We now complete the proof of Theorem 2.6, showing that if Conjecture 3.8 holds for all simple Schubert problems on $\mathbb{F}\ell(\alpha;n)$, then Conjecture 3.8 holds for all Schubert problems on $\mathbb{F}\ell(\beta,n)$, for any subsequence β of α. Here, we drop the claim of transversality in Conjecture 3.8. The proof will involve Schubert problems $w = (w_1, \dots, w_m)$ on $\mathbb{F}\ell(\alpha;n)$ such that $\pi(w) = (\pi(w_1), \dots, \pi(w_m))$ is a Schubert problem on $\mathbb{F}\ell(\beta;n)$, where $\pi\colon \mathbb{F}\ell(\alpha;n) \to \mathbb{F}\ell(\beta;n)$ is the projection map. When this happens and the problem w has non-zero degree, we say that the Schubert problem w is *fibred over* $\pi(w)$. Note that we do not require the two problems to have the same degree. While it is not the case that $\pi(w)$ is a Schubert problem on $\mathbb{F}\ell(\beta;n)$ whenever w is a Schubert problem on $\mathbb{F}\ell(\alpha;n)$, it turns out that for every Schubert problem v on $\mathbb{F}\ell(\beta;n)$, there are many Schubert problems w on $\mathbb{F}\ell(\alpha;n)$ which are fibred over v, and the degree of w is always a positive multiple of the degree of v. The geometry behind this is discussed, for instance, in [15].

Indeed, the fibre Y of the projection $\pi\colon \mathbb{F}\ell(\alpha;n) \to \mathbb{F}\ell(\beta;n)$ is a (product of) flag manifolds. The map $\pi\colon X_wF_\bullet \to X_{\pi(w)}F_\bullet$ is almost a fibre bundle. The fibre over a general point of $X_{\pi(w)}F_\bullet$ is a Schubert variety in Y whose indexing permutation is $\pi(w)^{-1}w$. Then if the flags are in general position, then π restricts to a fibration

$$\pi \;:\; X_{w_1}F^1_\bullet \cap \cdots \cap X_{w_m}F^m_\bullet \;\longrightarrow\; X_{\pi(w_1)}F^1_\bullet \cap \cdots \cap X_{\pi(w_m)}F^m_\bullet \tag{3.15}$$

with fibre the Schubert intersection in Y given by $(\pi(w_1)^{-1}w_1, \dots, \pi(w_m)^{-1}w_m)$.

When a problem w is fibred over a problem v, there may be conditions w_i of w such that $\pi(w_i) = \iota$, the identity permutation. This condition ι is *trivial* because $X_\iota = \mathbb{F}\ell(\beta;n)$. Two problems v and v' are *equivalent* if they differ only in trivial conditions.

The full statement of Theorem 2.6 is a consequence of the following result and the version when $\beta = \alpha$ already proven.

Theorem 3.16. *Suppose that β is a subsequence of α and that v is a simple Schubert problem for $\mathbb{F}\ell(\beta;n)$. Then there is a simple Schubert problem w for $\mathbb{F}\ell(\alpha;n)$ such that if Conjecture 3.8 holds for w, then it holds for v.*

Proof. Suppose that $w = (w_1, \dots, w_m)$ is a simple Schubert problem on $\mathbb{F}\ell(\alpha;n)$, and each w_i is a simple transposition of the form σ_{α_j}, for some j. Then

$$\pi(\sigma_{\alpha_j}) \;=\; \begin{cases} \sigma_{\alpha_j} & \text{if } \alpha_j \in \beta \\ \iota & \text{otherwise}\,. \end{cases}$$

It follows that $\pi(w)$ is a simple Schubert problem on $\mathbb{F}\ell(\beta;n)$ which involves some trivial Schubert varieties X_ι. As in the proof of Theorem 3.13, if $(t_1,\dots,t_m)$ is monotone for w, then it will be monotone for $\pi(w)$. Note that if $\pi(w_i)=\iota$, then the choice of the point t_i does not affect the Schubert intersection for $\pi(w)$.

The converse is also true. Let v be the Schubert problem $\pi(w)$, where we have dropped all of the trivial conditions ι. Any monotone choice of points for v may be extended to a monotone choice of points $(t_1,\dots,t_m)$ for w. We need only choose points t_i for those w_i such that $\pi(w)=\iota$ in a way to preserve monotonicity, which is easy.

Suppose now that v is a simple Schubert problem on $\mathbb{F}\ell(\beta;n)$. Then there is a simple Schubert problem w on $\mathbb{F}\ell(\alpha;n)$ which is fibred over v. Indeed, let Y be the flag manifold which is the fibre of the projection $\pi\colon \mathbb{F}\ell(\alpha;n)\to\mathbb{F}\ell(\beta;n)$. It suffices to add simple Schubert conditions to v coming from any simple Schubert problem on Y with degree >0. These added conditions w_i have descents in $\alpha\setminus\beta$, so the Schubert problems $\pi(w)$ and v are equivalent. Pick a monotone choice of points for v and, as explained in the previous paragraph, extend it to a monotone choice of points for w. If Conjecture 3.8 holds for w, then all the points in $X_{w_1}(t_1)\cap\cdots\cap X_{w_m}(t_m)$ are real. The map π (3.15) exhibits this as a surjection onto $X_{\pi(w_1)}(t_1)\cap\cdots\cap X_{\pi(w_m)}(t_m)$, which equals the corresponding intersection for the Schubert problem v and the original monotone choice of points. $\square$

Example 3.17. Theorem 3.16 involved one Schubert problem fibred over another. An example is provided by the Schubert problem $({\sigma_2}^4, (1245)^4)$ on $\mathbb{F}\ell(2,4;6)$, which has degree 6. As with the example in Section 2.3.5, this is fibred over the Schubert problem on $\mathrm{Gr}(4,6)$ involving the intersection of four Schubert varieties Ω_{1245} given by flags osculating the rational normal curve at the points corresponding to the conditions 1245. All three solution 4-planes K_1, K_2, and K_3 are real, and the fibre over K_i is the the problem of four 2-planes in K_i meeting four two planes that are the intersection of K_i with four 4-planes osculating the rational normal curve at the points corresponding to the conditions σ_2.

This problem in the fibre is not equivalent to an instance of the Shapiro conjecture for 2-planes in the 4-space K_i. If it were equivalent to an instance of the Shapiro conjecture, then all solutions for every necklace of Table 8 would be real, which is not the case. In the necklaces of Table 8, 2 represents the condition σ_2 and 4 represents the condition 1245.

3.5. **Discriminants.** Let $w=(w_1.\dots,w_m)$ be a Schubert problem on $\mathbb{F}\ell(\alpha;n)$. The *discriminant* $\Sigma\subset(\mathbb{P}^1)^m$ is the set of points $(t_1,\dots,t_m)$ where the intersection

$$X_{w_1}(t_1)\cap\cdots\cap X_{w_m}(t_m) \tag{3.18}$$

is not transverse. When $\Sigma\neq(\mathbb{P}^1)^m$, this is a hypersurface defined by the discriminant polynomial $\Delta_w(t_1,\dots,t_m)$, which is separately homogeneous in each homogeneous parameter t_i. For each of three Schubert problems, we will prove a weaker version of Conjecture 2.10 which implies Conjecture 2.2.

This version is weaker because we do not compute $\Delta_w(t_1,\dots,t_m)$, as that would be infeasible. Instead, we will fix three parameters, say $t_1=\infty$, $t_2=0$, and $t_3=1$ (or $t_3=-1$). Then we can carry out the computation in the local coordinates $\mathcal{M}_{w_1}$ for $X^\circ_{w_1}(\infty)$, or in local coordinates for the intersection of two cells $X^\circ_{w_1}(\infty)\cap X^\circ_{w_2}(0)$. In these

Necklace	Number of Real Solutions			
	0	2	4	6
22224444	0	0	0	100000
22242444	0	0	0	100000
22244244	0	0	0	100000
22442244	0	0	122	99878
22424244	0	12	3551	96437
22424424	0	105	8448	91447
24242424	0	1050	19964	78986
22422444	18	340	5147	94495

TABLE 8. The Schubert problem $({\sigma_2}^4, (1245)^4)$ on $\mathbb{F}\ell(2,4;6)$.

coordiantes, we generate the ideal defining the intersection (3.18), compute an eliminant $F(x;t)$ for one of our coordinates, and then compute its discriminant $\Delta_x(t)$.

If we specialize the parameters t_1, t_2, and t_3 to these fixed values, then $\Delta_w(t_1,\dots,t_m)$ will divide $\Delta_x(t)$, but there may be other factors in $\Delta_x(t)$. We minimized these extraneous factors by computing the greatest common divisor of these discriminants $\Delta_x(t)$ for each coordinate x. We also remove factors common to a leading term of any eliminant, as those correspond to solutions that are not on our chosen coordinate patch.

Theorem 3.19. *Conjecture* 2.10 *holds for the following Schubert problems.*

(1) $(\sigma_1, 4\,3\,1256, 13\,25\,46, 1256\,4\,3, \sigma_5)$ *on* $\mathbb{F}\ell(1,2,4,5;6)$. *This has* 2 *solutions.*
(2) $(24\,135, 13\,245, 134\,25, (124\,35)^2)$ *on* $\mathbb{F}\ell(2,3;5)$. *This has* 3 *solutions.*
(3) $(146\,2357, 135\,2467, 1246\,357, 1256\,347)$ *on* $\mathbb{F}\ell(3,4;7)$. *This has* 4 *solutions.*

Proof. (1) Consider the Schubert intersection

$$X_{\sigma_1}(t) \cap X_{4\,3\,1256}(\infty) \cap X_{13\,25\,46}(-1) \cap X_{1256\,4\,3}(0) \cap X_{\sigma_5}(s)\,,$$

on $\mathbb{F}\ell(1,2,4,5;6)$. Since these Schubert conditions have respective descent sets

$$\{1\},\ \{1,2\},\ \{2,4\},\ \{4,5\},\ \{5\}\,,$$

the set of monotone points is $\{(s,t) \mid 0 < s < t\}$. The discriminant we computed had two factors. One was s^6 and here is the other factor

$$\begin{aligned} &2500s^4t^4 + 18000s^3t^4 + 4000s^4t^3 + 50000s^2t^4 + 31100s^3t^3 + 2260s^4t^2 \\ &+ 64000st^4 + 91400s^2t^3 + 20040s^3t^2 + 480s^4t \\ &+ 32000t^4 + 122800st^3 + 63905s^2t^2 + 5550s^3t + 9s^4 \\ &+ 64000t^3 + 91400st^2 + 20040s^2t + 480s^3 \\ &+ 50000t^2 + 31100st + 2260s^2 + 18000t + 4000s + 2500\,. \end{aligned}$$

This is a positive sum of monomials and is thus positive when $0 < s, t$, which includes the set of monotone points.

$$\begin{aligned}
&800t^9 + 3600t^8(s-t) + 7744t^7(s-t)^2 + 10304t^6(s-t)^3 + 8736t^5(s-t)^4 + 4480t^4(s-t)^5\\
&+ 1248t^3(s-t)^6 + 144t^2(s-t)^7 + 5760t^8 + 23040t^7(s-t) + 45792t^6(s-t)^2 + 56736t^5(s-t)^3\\
&+ 43632t^4(s-t)^4 + 19584t^3(s-t)^5 + 4608t^2(s-t)^6 + 432t(s-t)^7 + 17888t^7 + 62608t^6(s-t)\\
&+ 114816t^5(s-t)^2 + 130520t^4(s-t)^3 + 87520t^3(s-t)^4 + 32064t^2(s-t)^5 + 5616t(s-t)^6\\
&+ 324(s-t)^7 + 31712t^6 + 95136t^5(s-t) + 161496t^4(s-t)^2 + 164432t^3(s-t)^3\\
&+ 90048t^2(s-t)^4 + 23688t(s-t)^5 + 2268(s-t)^6 + 35456t^5 + 88640t^4(s-t)\\
&+ 141256t^3(s-t)^2 + 123244t^2(s-t)^3 + 48726t(s-t)^4 + 6777(s-t)^5 + 25376t^4\\
&+ 50752t^3(s-t) + 79184t^2(s-t)^2 + 53808t(s-t)^3 + 11394(s-t)^4 + 10752t^3\\
&+ 16128t^2(s-t) + 27264t(s-t)^2 + 10944(s-t)^3 + 2048t^2 + 2048t(s-t) + 4608(s-t)^2\ .
\end{aligned}$$

FIGURE 5. A discriminant.

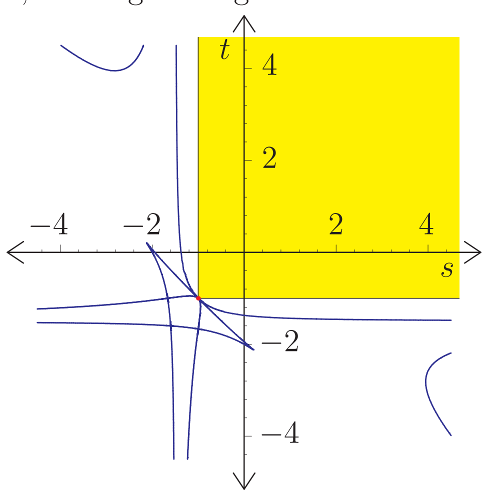
(2) Consider the Schubert intersection

$$X_{24\,135}(\infty) \cap X_{13\,245}(-1) \cap X_{134\,25}(0) \cap X_{124\,35}(s) \cap X_{124\,35}(t)$$

on the flag variety $\mathbb{F}\ell(2,3;5)$. Since these Schubert conditions have respective descents at $2, 2, 3, 3, 3$, the set of monotone points is

$$\{(s,t) \mid -1 < s,t,\ \ s,t \neq 0,\ \text{and}\ s \neq t\}\,. \tag{3.20}$$

We display the discriminant, shading the region with monotone points.



In Figure 5, we write this discriminant in terms of t and $s-t$, whose positivity defines the region where $0 < s < t$, a subset of the set of monotone points. This discriminant is a positive linear combination of 49 homogeneous monomials of degree 9 in the terms t and $s-t$, and is thus positive on the region defined by $0 < s < t$. There is a similar positive expression for the discriminant in terms of $1+t$ and s, and another in terms of $1+t$, $s-t$, and $-s$. Together with the expression in Figure 5, these show that the discriminant is positive on the set $-1 < s < t$ with $s,t \neq 0$. Since the discriminant is symmetric in s

and t, the symmetric counterpart of these three expressions shows that the discriminant is positive on the set (3.20) of monotone points.

(3) Consider the Schubert intersection

$$X_{146\,2357}(\infty) \cap X_{135\,2467}(0) \cap X_{1246\,357}(s) \cap X_{1256\,347}(1)$$

on the flag variety $\mathbb{F}\ell(3,4;7)$. Since the Schubert conditions have descents $3,3,4,4$, the pointss are monotone when $0<s$. Removing factors of s and $1+s$ from the discriminant, we obtain

$$\begin{aligned} &3515625 + 45243750s + 221792500s^2 + 565872594s^3 + 777678231s^4 \\ &+ 1273923370s^5 + 932192307s^6 + 909742337s^{10} + 1560886138s^{11} \\ &+ 867109112s^{12} + 367416324s^{13} + 114976512s^{14} + 13608000s^{15} + 648000s^{16} \\ &+ \big(42966406s^3 + 352158344s^4 + 135425340s^5\big)(1-s^4)^2 \ , \end{aligned}$$

which is obviously positive when $0<s$. □

Remark 3.21. The first discriminant we computed, for the Schubert intersection

$$X_{\sigma_1}(t) \cap X_{4\,3\,1256}(\infty) \cap X_{13\,25\,46}(-1) \cap X_{1256\,4\,3}(0) \cap X_{\sigma_5}(s)\,, \tag{3.22}$$

was positive on more than just the monotone region.

Figure 6 compares the table for this Schubert problem with a plot of the discriminant, which proves that 9 of the 12 necklaces will give only real solutions. Indeed, the

Label	Necklace	0	2
I	ABCst	0	100000
II	ABCts	0	100000
III	ABstC	0	100000
IV	ABtsC	0	100000
V	AtsBC	0	100000
VI	AstBC	0	100000
VII	ABsCt	0	100000
VIII	AtBCs	0	100000
IX	AtBsC	0	100000
X	ABtCs	24976	75024
XI	AsBCt	26065	73935
XII	AsBtC	38023	61977

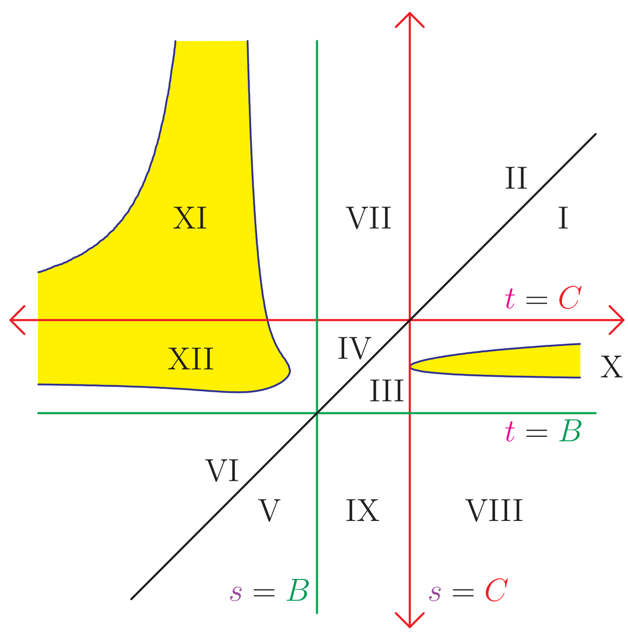


FIGURE 6. The Schubert problem $(\sigma_1, 4\,3\,1256, 13\,25\,46, 1256\,4\,3, \sigma_5)$ on $\mathbb{F}\ell(1,2,4,5;6)$ and its discriminant.

shaded region is where the discriminant is negative. The (s,t)-plane is divided into 12 regions by the lines $s=t$ and $s,t=0,-1$, which are points that cannot be used in the intersection (3.22). Each region corresponds to a necklace, and is labeled by the row of its corresponding necklace. For the necklaces, we use t, A, B, C, and s to denote the conditions σ_1, $4\,3\,1256$, $13\,25\,46$, $1256\,4\,3$, and σ_5, respectively.

4. Methods

The *raison d'être* for this paper is our computer experimentation investigating the number of real points in Schubert intersections of the form

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,, \tag{4.1}$$

for Schubert problems $(w_1, w_2, \ldots, w_m)$ on small flag manifolds. We determined this number for 520,420,135 different intersections involving 1126 different Schubert problems on 29 different flag manifolds. This used 15.76 gigahertz-years of computer time.

Table 9 shows the effort devoted to studying the three main conjectures: The Shapiro Conjecture for Grassmannians (Conjecture 1.3), our Monotone Conjecture for Grassmannian Schubert problems (Conjecture 2.2), and the Refined Monotone Conjecture (Conjecture 3.8). Since these are in increasing order of generality, each of the last two rows of Table 9 only shows the extra effort devoted to the corresponding conjecture. The numbers

	Number of Problems	Number of Intersections	gigahertz years
Conjecture 1.3	212	132,919,238	3.57
Conjecture 2.2	376	25,524,191	1.23
Conjecture 3.8	201	7,223,660	0.77

Table 9. Resources devoted to the conjectures.

for the last two conjectures are only a small fraction of the total effort expended in this experimentation. This is because only a small fraction of necklaces are monotone.

A significant part of our investigation was devoted to the Shapiro Conjecture for Grassmannians (Conjecture 1.3), for $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(4,8)$. While this conjecture had been studied before [24], the scope of previous experiments was limited.

Section 4.1 explains how we determined the number of real solutions in an intersection (4.1). Section 4.2 describes how we investigated such intersections for many necklaces and choices of points for a single Schubert problem. Section 4.3 discusses the design of the experiment, that is, how we chose which Schubert problems to investigate.

4.1. Computation of a single Schubert intersection (4.1).

All computations were done on Intel processors running Linux, using the computer algebra systems Singular `2-0-5` [8] and Maple, which were called from `bash` shell scripts. Maple managed the data, created the Singular scripts, and counted the real solutions to univariate eliminants.

To study a Schubert intersection (4.1), we generated the ideal of the intersection in local coordinates $\mathcal{M}_{w_1}$ parametrizing the Schubert cell $X^\circ_{w_1}(\infty)$. For this, we fixed $t_1 = \infty$, and the other points $t_2, \ldots, t_m$ were rational numbers, and the ideal was generated by the equations for each Schubert variety $X_{w_i}(t_i)$ as described in Section 1.1 and in Section 1.2

(where the flags $F_\bullet(t_i)$ were described). Because Gröbner basis computation is extremely sensitive to the number of variables, the first Schubert condition w_1 was chosen to minimize the number of coordinates in the parametrization $\mathcal{M}_{w_1}$ of the Schubert cell $X^\circ_{w_1}(\infty)$.

Singular computed a degree reverse lexicographic Gröbner basis for this ideal and then used the FGLM algorithm [6] to compute a square-free univariate eliminant with degree equal to the degree of the Schubert problem. This guaranteed that the original intersection was transverse and that its number of real points is equal to the number of real roots of the eliminant (see the discussion in [26, §2.2]). This number of real roots was computed using the `realroot` command in Maple. When such an eliminant could not be computed, data describing the intersection were set aside and later studied by hand.

4.2. **Investigation of a single Schubert problem.** For a given Schubert problem $(w_1, \ldots, w_m)$, we determined the number of real points in many different Schubert intersections of the form (4.1). Once a problem was selected, data necessary for the experimentation were precomputed and stored in a data file. These data included a list L of permutations of the numbers $\{2, \ldots, m\}$ and a set S of rational numbers. The list L typically consisted of one permutation representing each necklace we decided to investigate. This data file was updated throughout the computation as it also recorded the numbers of real solutions found for the different necklaces and for different choices of points.

Most Schubert problems were run on a single computer. The actual computation was organized by a shell script, whose main part was a loop. In each iteration, the loop variable was used as a seed for Maple's random-number generator to select a random subset $t_2, \ldots, t_m$ of the points from S, which were ordered so that $t_2 < \cdots < t_m$. For each permutation σ of L, the number of real points in the intersection

$$X^\circ_{w_1}(\infty) \cap X_{w_2}(t_{\sigma(2)}) \cap X_{w_3}(t_{\sigma(3)}) \cap \cdots \cap X_{w_m}(t_{\sigma(m)}) \tag{4.2}$$

was determined and included in the data file. The data file also kept track of the CPU time used in the computation, and recorded the average size of the univariate eliminants. The number of iterations of the shell script depended upon our interest in the problem and the computational cost.

After the computations were completed for a given Schubert problem, the data file was used to generate a web page which displayed information from the experimentation on that Schubert problem. Figure 7 illustrates a typical such page. This web page has a key in the form of a table with one row for each Schubert condition. Each row shows the condition as a permutation, then in a shorthand that is well-suited to Grassmannian conditions—the letter indicates which member of the flag it is imposed upon and the partition index indicates the corresponding Schubert condition on the Grassmannian. Next is the symbol for that condition used when listing the necklaces, and finally its codimension. The figure under Point Selection shows the positions of the points in S on $\mathbb{RP}^1$, represented as a circle where the point at the top is ∞. This web page also records the total computation time, the machine used (Noether is a computer owned by Sarah Witherspoon), the total number of polynomial systems solved, and the size of a typical eliminant.

This web page for the problem $W_{\square\square\square}(X_{\square})^2(Y_{\square})^4$ is linked to web pages of problems fibred over $W_{\square\square\square}(X_{\square})^2(Y_{\square})^4$ and to web pages of problems over which $W_{\square\square\square}(X_{\square})^2(Y_{\square})^4$ is

Enumerative problem $W_{\square\square\square}(X_{\square})^2(Y_{\square})^4 = 7$ on $\mathbb{F}\ell(1,2,3;5)$

Up

Experimental data

	Number of Real Solutions			
Necklace	**1**	**3**	**5**	**7**
abbcccc	0	0	0	25000
abccccb	0	0	0	25000
accbbbb	0	89	10500	14411
acbbccc	0	2374	5740	16886
abccbcc	0	2560	13204	9236
abcccbc	0	4456	9753	10791
aabccbc	29	2571	14627	7773
abcbccc	1120	5364	9633	8883
acbcbcc	3446	5566	9132	6856

Related Problems

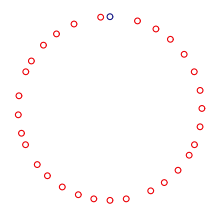
Projections

Variety	Problem	#
Fl(2,3;5)	$(X_{\square})^2 X_{\square\square}(Y_{\square})^4$	7

Problems fibred over $W_{\square\square\square}(X_{\square})^2(Y_{\square})^4$

Variety	Problem	#
Fl(1,2,3,4;5)	$W_{\square\square\square}(X_{\square})^2(Y_{\square})^4 Z_{\square}$	7
Fl(1,2,3,4;5)	$W_{\square\square\square}(X_{\square})^2(Y_{\square})^3 Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	7
Fl(1,2,3,4;5)	$A_{4125}(X_{\square})^2(Y_{\square})^4$	7

Point Selection



Key			
Condition	**Name**	**Symbol**	**Codimension**
412	$W_{\square\square\square}$	**a**	3
132	$X_{\square}$	**b**	1
124	$Y_{\square}$	**c**	1

Total time of computation: 27,491.26 GHz-seconds or 7.64 GHz-hours on Noether

225 000 Polynomial systems solved

The coefficients of a typical eliminant had 29 digits.
The typical eliminant had size 271 bytes.

This table automatically generated from the data in This File using
This Maple Script

Created: Fri Jul 15 15:42:38 CDT 2005

FIGURE 7. Web page for the problem $(412, (132)^2, (124)^4)$ on $\mathbb{F}\ell(1,2,3;5)$.

fibred (called Projections). It is also linked to the data file and to the Maple script used to generate the web page. At its top is a link Up to the web page for the flag variety $\mathbb{F}\ell(1,2,3;5)$. That page lists all 163 Schubert problems we studied on $\mathbb{F}\ell(1,2,3;5)$, is linked to the other 8 flag varieties in 5-space that we investigated, and to a page with information about the 29 different flag varieties in our investigation.

This archive of our data is part of a web page containing additional information about this project, which is found at `www.math.tamu.edu/~sottile/pages/Flags/`, the page displayed has further extension `Data/F1235/W3Xe2Ye4.7.html`. Subsequent addresses will give only the extension from `.../Flags/Data/`.

4.3. **Design of experiments.** While we investigated many Schubert problems on many small flag manifolds, by no means did we study all Schubert problems on these flag manifolds. We did investigate all Schubert problems on the manifolds of flags in $\mathbb{C}^4$, and all with degree at least 3 on $\mathbb{F}\ell(1,2,3;5)$, $\mathbb{F}\ell(1,2,4;5)$, $\mathbb{F}\ell(1,2;5)$, $\mathbb{F}\ell(1,3;5)$, $\mathbb{F}\ell(2,3;5)$, $\mathbb{F}\ell(2,4;5)$, $\mathbb{F}\ell(3,4;5)$, and $\mathrm{Gr}(3,6)$. Only a small fraction of feasible Schubert problems were investigated on the other 18 flag manifolds.

There were limitations of resources which made choices necessary. For example, the complexity of Gröbner basis computation limited us to Schubert problems of low degree (typically fewer than 20 solutions). For the computations on Grassmannians, a more advantageous choice of local coordinates was possible, which allowed significantly larger problems—we studied one problem on $\mathrm{Gr}(3,7)$ with 91 solutions[1].

Many Schubert problems had literally thousands of necklaces, such as the problem of Table 3 with 11,352 neckalces. A systematic study of all necklaces for such a problem would be infeasible and the data would be incomprehensible. We did consider all 1272 necklaces for one such problem[2]. Limiting our investigation to problems of small degree and with few necklaces would still have been infeasible, as there are many thousands of such smaller Schubert problems on some of these flag manifolds.

On the flag manifolds for which it was impossible to investigate all Schubert problems, we studied most feasible Grassmannian Schubert problems, as well as many related to these Grassmannian problems through projection, lifting, fibration, and the notion of child problems as discussed in Sections 3.1 and 3.4. We looked at some with potentially interesting geometry such as the problem of Section 2.3.5. We also selected many problems completely at random, intending to sample the range of possibilities.

Table 10 lists the Schubert problems discussed here, their associated web pages, and the resources expended on each.

5. Conclusion and Future Work

We presented a geometrically vivid example of the failure of the Shapiro Conjecture for Schubert intersections given by osculating flags on flag manifolds, and presented a refinement of the conjecture for flag varieties. Significant evidence, both theoretical and

[1] `F37/We7W2W21.91.html`

[2] `F12456/Ve2We2W32Ye3Ze2.4.html`

Location	Web Page	CPU
Table 1	`F235/Xe4Ye4.12.html`	213.38 GHz-days
Table 2	`F12345/We2Xe3Ye3Ze2.12.html`	47.61 GHz-days
Table 3	`F123456/Ve2We2XX321Ye2Ze2.8.html`	5.25 GHz-days
Table 4	`F347/WW31e2Xe2X211.10.html`	63.43 GHz-days
Table 5	`F12345/A1325e2A2143e3.7.html`	1.94 GHz-days
Table 6	`F1356/A21436e2A31526Xe2.8.html`	12.84 GHz-days
Table 7	`F2346/A1432A1254We2X21Ye2.12.html`	61.86 GHz-days
Table 8	`F246/We4Y11e4.6.html`	13.57 GHz-days
Figure 6	`F12456/A13254A43125A12564VZ.2.html`	1.31 GHz-days

Table 10. CPU time used for computations shown here

experimental, was presented in support of this refinement. Several new phenomena discovered in this experimentation were presented.

The proof of the conjecture for certain two-step flag manifolds by Eremenko *et al.* leads to an extension concerning secant flags. The further investigation of this secant flag conjecture is a worthwhile future project.

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