

# REAL HESSIAN CURVES

ADRIANA ORTIZ-RODRÍGUEZ AND FRANK SOTTILE

ABSTRACT. We give some real polynomials in two variables of degrees 4, 5, and 6 whose hessian curves have more connected components than had been known previously. In particular, we give a quartic polynomial whose hessian curve has 4 compact connected components (ovals), a quintic whose hessian curve has 8 ovals, and a sextic whose hessian curve has 11 ovals.

## INTRODUCTION

The *parabolic curve* on a generic smooth surface  $S$  embedded in three-dimensional Euclidean space consists of the points where  $S$  has zero Gaussian curvature. It separates *elliptic points* (where the curvature is positive) from *hyperbolic points* (where the curvature is negative). These notions are well-defined for surfaces embedded in affine or even projective space, as the sign of Gaussian curvature is invariant under affine transformations.

If the surface  $S$  is expressed locally as the graph  $z = f(x, y)$  of a smooth function  $f$ , then the sign of its hessian determinant

$$\text{Hess}(f) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2,$$

equals the sign of its curvature at the corresponding point. Thus the parabolic curve is the image under  $f$  of its *hessian curve*, which is defined by  $\text{Hess}(f) = 0$ . When the surface  $S$  is the graph of a polynomial  $f \in \mathbb{R}[x, y]$ , this local description is global, and so questions about the disposition of the parabolic curve on  $S$  are equivalent to the same questions about the hessian curve in  $\mathbb{R}^2$ .

Suppose that  $d$  is even. Harnack proved [3] that a smooth plane curve of degree  $d$  has at most  $1 + \binom{d-1}{2}$  connected components in  $\mathbb{RP}^2$ . This is also the bound for the number of components of a compact curve in  $\mathbb{R}^2$  of degree  $d$ . A non-compact curve in  $\mathbb{R}^2$  of degree  $d$  can have at most  $\binom{d-1}{2}$  bounded components (*ovals*) and  $d$  unbounded components. These unbounded components come from the intersection of the corresponding curve in  $\mathbb{RP}^2$  with the line at infinity. Harnack constructed a curve in  $\mathbb{RP}^2$  of degree  $d$  with  $1 + \binom{d-1}{2}$  components which has one component meeting the line at infinity in  $d$  points. This Harnack curve shows that the bound for non-compact curves in  $\mathbb{R}^2$  is attained, and choosing a different line at infinity shows that the bound for compact curves in  $\mathbb{R}^2$  is also attained.

We are interested in the possible number and disposition of the components of the hessian curve in  $\mathbb{R}^2$  of a polynomial  $f \in \mathbb{R}[x, y]$  of degree  $n$ . This is problem 2001-1 in the list of Arnold's problems [1], attributed to A. Ortiz-Rodríguez. See also the discussion of related problems 2000-1, 2000-2, 2001-1, and 2002-1. The hessian of  $f$  has degree at most  $2n-4$ . By

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Harnack's Theorem, a compact hessian curve has at most  $(2n-5)(n-3)+1$  ovals and a non-compact hessian curve has at most  $(2n-5)(n-3)$  ovals and  $2n-4$  unbounded components.

While we know of no additional restrictions on hessian curves, we are not assured that all possible configurations are achieved by Hessians. When  $n$  is at least 4, simple parameter counting shows that not all polynomials of degree  $2n-4$  arise as Hessians of polynomials of degree  $n$ . The placement of the set of hessian curves among all curves of degree  $2n-4$  may restrict the possible configurations of hessian curves in  $\mathbb{R}^2$ . For example, a simple calculation shows that

$$\text{Hess}(f) = \left( \frac{(f_{xx} + f_{yy})}{2} \right)^2 - \left( \frac{(f_{xx} - f_{yy})}{2} \right)^2 - f_{xy}^2.$$

Thus the hessian of a polynomial is a linear combination of 3 squares, which shows that the Hessians lie in the second secant variety to the Veronese embedding of polynomials of degree  $n-2$  in polynomials of degree  $2n-4$  (the Veronese consists of the perfect squares).

We also know of no general techniques for constructing hessian curves with a prescribed configuration. One of us (Ortiz-Rodríguez) investigated this question [4, 5] and constructed polynomials  $f \in \mathbb{R}[x, y]$  of degree  $n$  whose Hessians had  $\binom{n-1}{2}$  ovals in  $\mathbb{R}^2$ . When  $n$  is 4, 5, and 6, these numbers are 3, 6, and 10, respectively. We do not know if it is possible for a hessian curve to achieve the Harnack bound, or more generally, which configurations are possible for hessian curves.

Here, we present a quartic polynomial  $f$  whose hessian achieves the Harnack bound of 4 ovals, a quintic whose hessian has 8 ovals, a sextic whose hessian has 11 ovals, as well as examples of non-compact hessian curves of quartics, quintics, and sextics. These examples show that hessian curves can have more ovals than was previously known. They were found in a computer search, using the software Maple.

Our method was to generate a random polynomial, compute its hessian, and then compute an upper bound on its number of ovals in  $\mathbb{RP}^2$ , sometimes also screening for the number of unbounded components in  $\mathbb{R}^2$ . This upper bound was one-half the minimum number of real critical points of a projection to one of the axes, as each oval in  $\mathbb{RP}^2$  contributes at least two critical points to the projection. We separately investigated compact hessian curves of sextics. Polynomials whose upper bound for ovals was at least 4, 8, and 11 (for quartics, quintics, and sextics, respectively) were saved for further study. The further investigation largely involved viewing pictures in  $\mathbb{R}^2$  of these potentially interesting Hessians. In all, only a few hundred polynomials warranted such further scrutiny.

We examined the Hessians of 150 million quartics, 40 million each of quintics and sextics, and over 200 million sextics with compact Hessians (the different protocol of pre-screening for compactness allowed a greater number to be examined). This required 628 days of CPU time on several computers, most of which were running Linux on Intel Pentium processors with speeds between 1.8 and 3 GHz. We did not find a quartic whose hessian had 3 ovals and 4 unbounded components, nor a quintic whose hessian had more than 8 ovals in  $\mathbb{RP}^2$ , nor a sextic whose hessian had more than 11 ovals in  $\mathbb{RP}^2$ . (The examples we give at the end with 12 ovals in  $\mathbb{RP}^2$  are perturbations of a curve we found with 11 ovals.) This suggests that it may not be possible for hessian curves in  $\mathbb{R}^2$  to achieve the Harnack bounds. Further pictures and computer code are at the web page<sup>1</sup>.

Tables 1 and 2 summarize this discussion concerning the number of components of hessian curves. The pairs  $(o, u)$  in Table 2 refer to ovals and unbounded components, respectively.

<sup>1</sup>[www.math.tamu.edu/~sottile/stories/Hessian/index.html](http://www.math.tamu.edu/~sottile/stories/Hessian/index.html).

Degree of $f$	$n$	3	4	5	6	7
Degree of hessian	$2n-4$	2	4	6	8	10
Harnack bound for hessian	$(2n-5)(n-3) + 1$	1	4	11	22	37
Ortiz Hessians [4, 5]	$(n-1)(n-2)/2$	1	3	6	10	15
New examples			4	8	11	—

TABLE 1. Ovals of compact hessian curves.

Degree of $f$	$n$	3	4	5	6
Degree of hessian	$2n-4$	2	4	6	8
Harnack bound	$((2n-5)(n-3), 2n-4)$	(0,2)	(3,4)	(10,6)	(21,8)
New examples			(2,4)	(6,4)	(10,4)
			(3,2)	(7,2)	(11,2)

TABLE 2. Configurations of non-compact Hessians.

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### HESSIAN CURVES WITH MANY OVALS

We begin with the following observation about hessian polynomials.

**Proposition 1.** *A polynomial  $h(x, y)$  is a hessian of some polynomial  $f$  if and only if there exist polynomials  $p, q, r$  such that  $p_y = q_x$ ,  $q_y = r_x$ , and  $h = pr - q^2$ .*

*Proof.* If  $h$  is the hessian of  $f$ , then  $h = f_{xx}f_{yy} - f_{xy}^2$ , and  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  satisfy these conditions. Conversely, if  $p$ ,  $q$ , and  $r$  satisfy the conditions, then elementary integral calculus gives polynomials  $s$  and  $t$  such that  $s_x = p$ ,  $s_y = q$ ,  $t_x = q$ , and  $t_y = r$ . Since  $s_y = t_x$ , there is a polynomial  $f$  with  $f_x = s$  and  $f_y = t$ , and thus  $h$  is the hessian of  $f$ .  $\square$

**Theorem 2.** *There exists a real polynomial of degree 4 in two variables whose hessian curve is smooth, compact, and consists of exactly four ovals.*

*Proof.* Let  $f$  be the polynomial

$$-2y^2 + 2xy + 12x^2 + 10y^3 + 3xy^2 - 10x^2y - 13x^3 - 11y^4 + 6xy^3 + 9x^2y^2 - 2x^3y - x^4.$$

If we divide its hessian by  $-4$ , we obtain the polynomial

$$h := 25 - 134x - 374y + 91x^2 + 948xy + 1137y^2 + 429x^3 + 612x^2y - 2313xy^2 - 876y^3 + 63x^4 + 54x^3y - 99x^2y^2 - 234xy^3 + 675y^4.$$

We claim that the hessian curve,  $h(x, y) = 0$ , is a compact smooth curve in  $\mathbb{R}^2$  with exactly 4 connected components. A visual proof is provided by the picture of the hessian curve in Figure 1. This was drawn by Maple using its `implicitplot` function with  $200 \times 200$  grid. While this is sufficient to verify our claim, we provide alternative ad hoc arguments.

We compute the values of the hessian at the four points inside each oval of Figure 1

$$h(-2, 0) = -7068, \quad h(0, \frac{1}{5}) = -\frac{5124}{125}, \quad h(2, 2) = -8508, \quad \text{and} \quad h(2, -1) = -6828.$$

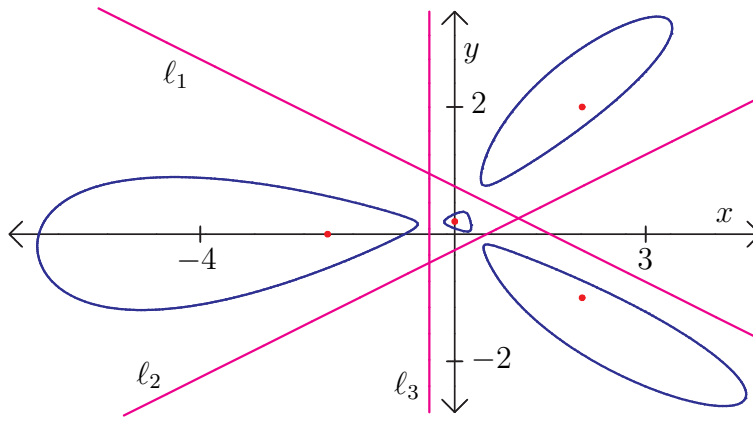


FIGURE 1. Quartic hessian curve with 4 compact components

Next, we shall prove that  $h$  is positive on the line  $\ell_\infty$  at infinity and on three lines of Figure 1,

$$\ell_1 : y = \frac{3}{4} - \frac{x}{2}, \quad \ell_2 : y = \frac{x}{2} - \frac{1}{4}, \quad \text{and} \quad \ell_3 : x = -\frac{2}{5}.$$

The lines  $\ell_\infty$ ,  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  divide  $\mathbb{RP}^2$  into 7 components. Since  $h$  is positive on these lines but is negative at the four points  $(-2, 0)$ ,  $(0, \frac{1}{5})$ ,  $(2, 2)$ , and  $(2, -1)$ , which lie in different regions, the hessian curve  $h = 0$  is compact and has at least one 1-dimensional component in each region surrounding one of the four points. Since 4 is the maximum number of one-dimensional connected components of a quartic, and such quartics are smooth, we deduce that the hessian curve is smooth, compact, and consists of exactly four ovals.

For the claim about positivity, note that  $h$  contains the monomial term  $63x^4$ , and so it is positive at one point on the line at infinity. We show that  $h$  does not vanish on any of these four lines, which implies our claim about positivity. For this, we invoke a classical characterization of when a univariate quartic has no real zeroes. References may be found, for example in [2, §71].

Given a univariate quartic polynomial of the form

$$z^4 + 4\alpha z^3 + \beta z^2 + \gamma z + \delta,$$

linear substitution of  $(z - \alpha)$  for  $z$  gives the reduced quartic

$$z^4 + az^2 + bz + c,$$

where  $a = \beta - 6\alpha^2$ ,  $b = \gamma - 2\alpha\beta + 8\alpha^3$ , and  $d = \delta - \alpha\gamma + \alpha^2\beta - 3\alpha^4$ . The discriminant of this reduced quartic is

$$\Delta := -4a^3b^2 - 27b^4 + 16a^4c - 128a^2c^2 + 144ab^2c + 256c^3.$$

This criterion also uses the polynomial

$$L := 2a(a^2 - 4c) + 9b^2.$$

Then the quartic has no real zeroes if and only if


$$(1) \quad \Delta > 0 \quad \text{and either} \quad a \geq 0 \quad \text{or} \quad L \geq 0.$$

Homogenizing  $h$ , restricting it to the line at infinity, substituting  $y = 1$ , and dividing by 9 gives the quartic

$$q_\infty := 7x^4 + 6x^3 - 11x^2 - 26x + 75.$$

Restricting  $h$  to the lines  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  and clearing denominators gives

$$\begin{aligned} q_1 &:= 21168x^4 - 157632x^3 + 592264x^2 - 337648x + 58387, \\ q_2 &:= 20016x^4 + 4608x^3 + 377320x^2 - 278112x + 52707, \text{ and} \\ q_3 &:= 421875y^4 - 489000y^3 + 1278975y^2 - 411710y + 42073. \end{aligned}$$

These satisfy the criterion (1) to have no real zeroes, as may be seen from Table 3, where we give the values of  $\Delta$ ,  $L$ , and  $a$ , for each of these polynomials. 

Polynomial	$\Delta$	$L$	$a$
$q_\infty$	$\frac{5025022208}{16807}$	$\frac{564896}{2401}$	$\frac{-181}{98}$
$q_1$	$\frac{105415059013155058653376}{198607342807439307}$	$\frac{3692894126604316}{340405734249}$	$\frac{931453}{129654}$
$q_2$	$\frac{34807374069358185363904}{141964610099247963}$	$\frac{4123100447100116}{272136458889}$	$\frac{6549023}{347778}$
$q_3$	$\frac{10042565821320692218681168}{855261504650115966796875}$	$\frac{1376823939540422}{40045166015625}$	$\frac{1066423}{421875}$

TABLE 3. Values of  $\Delta$ ,  $L$ , and  $a$ .

Each of the remaining curves we discuss is smooth, each oval has exactly two vertical and two horizontal tangents, and each unbounded component has exactly one vertical and one horizontal tangent. These claims are best verified symbolically. For each, we give the polynomial  $f$  and display a picture of the hessian curve, drawn with the `implicitplot` function of Maple. These were rendered, at least locally, with a grid size sufficiently small to separate the tangents, and therefore provide a faithful picture of the hessian curves as curves in  $\mathbb{R}^2$ .

Figure 2(a) displays the hessian curve of the quartic

$$22x^2 + 36xy + 24y^2 - 80x^3 - 10x^2y + 71xy^2 + 39y^3 + 15x^4 + 4x^3y - 3x^2y^2 - 21xy^3 - 17y^4,$$

which has 3 ovals and 2 unbounded components. Figure 2(b) displays the hessian curve of the quartic

$$-70x^2 - 35xy - 2y^2 - 93x^3 - 14x^2y + 41xy^2 - 70y^3 + 31x^4 + 7x^3y - 30x^2y^2 + 37xy^3 + 91y^4,$$

which has 2 ovals and 4 unbounded components. While we have generated and checked 150

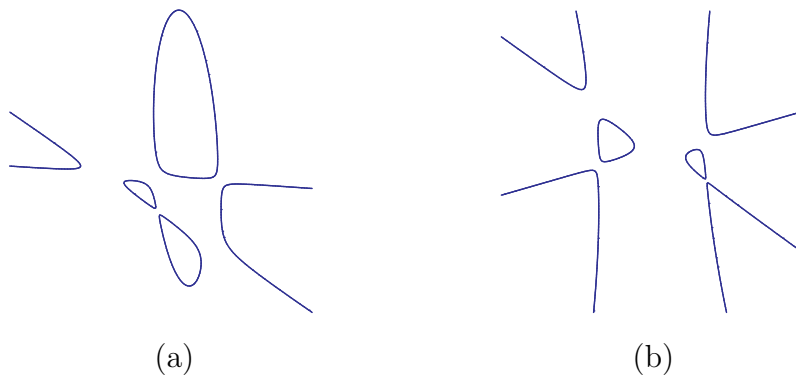


FIGURE 2. Hessians of quartics

million quartics, we did not find one whose hessian curve achieves the Harnack bound of 3 ovals and 4 unbounded components.

Figure 3(a) displays the hessian curve of the quintic

$$4y^2 + xy - 6x^2 - 25y^3 + 24xy^2 + 15x^2y - 33x^3 + y^4 - 3xy^3 + 15x^2y^2 - 19x^3y - 26x^4 + 33y^5 - 2xy^4 - 23x^2y^3 - 30x^3y^2 - 26x^4y + 31x^5 ,$$

which is compact with 8 ovals.

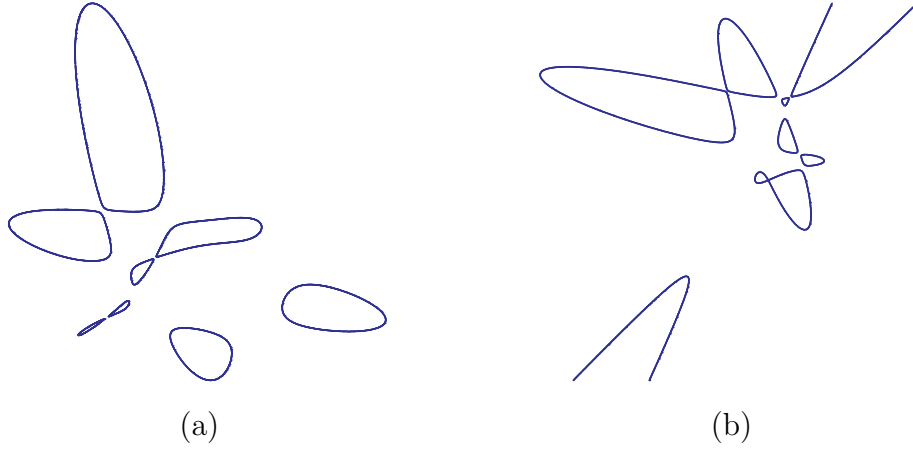


FIGURE 3. Hessians of quintics

Figure 3(b) displays the hessian curve of the quintic

$$- 54y^2 - 103xy - 26x^2 - 88y^3 + 45xy^2 + 91x^2y - 96x^3 - 12y^4 + 43xy^3 + 6x^2y^2 + 11x^3y + 49x^4 + 22y^5 - 20xy^4 - 38x^2y^3 - 14x^3y^2 + 45x^4y + 76x^5 ,$$

which has 7 ovals and 2 unbounded components.

Figure 4 displays the hessian curve of the quintic

$$60y^2 + 21xy + 76x^2 + 95y^3 - 18xy^2 - 79x^2y + 88x^3 - 25y^4 - 22xy^3 + 50x^2y^2 - 9x^3y - 5x^4 - 57y^5 - 50xy^4 + 21x^2y^3 + 87x^3y^2 + 35x^4y - 56x^5 ,$$

which has 6 ovals and 4 unbounded components. The boxed region on the left has been expanded in the picture on the right.

These quintics all have 8 ovals in  $\mathbb{RP}^2$ . While we have generated and checked 40 million quintics, we did not find any with more ovals.

Figure 5 displays the hessian curve of the sextic

$$45y^2 - 47xy - 30x^2 + 96y^3 - xy^2 + 8x^2y + 54x^3 - 96y^4 - 64xy^3 - 50x^2y^2 - 33x^3y + 91x^4 - 100y^5 + 84xy^4 - 43x^3y^2 + 66x^4y - 58x^5 + 70y^6 + 90xy^5 - 28x^2y^4 - 53x^3y^3 + 43x^4y^2 + 36x^5y - 38x^6 ,$$

which has 11 ovals. The boxed region on the left has been expanded in the picture on the right.

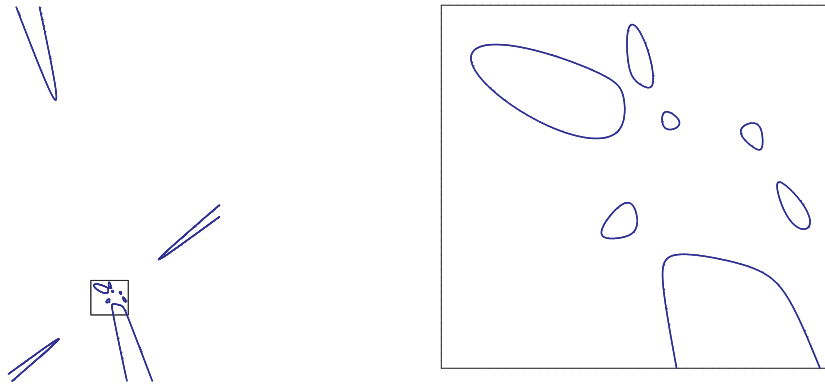


FIGURE 4. Hessian of a quintic with 6 ovals and 4 unbounded components

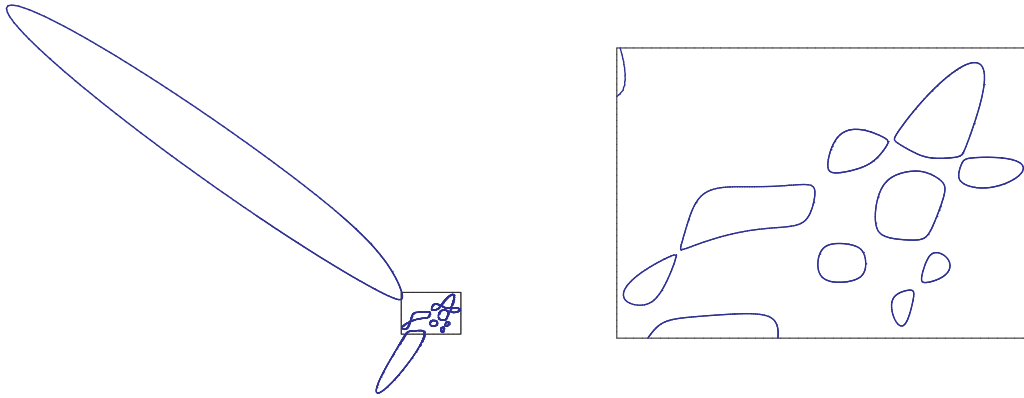


FIGURE 5. Hessian of a sextic with 11 ovals.

Figure 6(a) displays the hessian curve of the sextic

$$\begin{aligned}
 & - 53y^2 - 31xy + 59x^2 - 79y^3 + 82xy^2 - 52x^2y + 22x^3 \\
 & + 75y^4 - 27xy^3 + 63x^2y^2 - 85x^3y - 89x^4 \\
 & + 80y^5 + 27xy^4 - 69x^2y^3 + 17x^3y^2 - 7x^4y - 43x^5 \\
 & - 25y^6 + 17xy^5 + 27x^2y^4 - 55x^3y^3 - 37x^4y^2 + 59x^5y + 45x^6 ,
 \end{aligned}$$

which has 11 ovals and 2 unbounded components.

Figure 6(b) displays the hessian curve of the sextic

$$\begin{aligned}
 & - 80y^2 - 46xy + 89x^2 - 118y^3 + 123xy^2 - 78x^2y + 33x^3 \\
 & + 113y^4 - 40xy^3 + 94x^2y^2 - 128x^3y - 133x^4 \\
 & + 120y^5 + 40xy^4 - 104x^2y^3 + 25x^3y^2 - 10x^4y - 64x^5 \\
 & - 37y^6 + 25xy^5 + 40x^2y^4 - 82x^3y^3 - 56x^4y^2 + 89x^5y + 67x^6 ,
 \end{aligned}$$

which has 10 ovals and 4 unbounded components. Both hessian curves have 12 ovals in  $\mathbb{RP}^2$ .

Despite examining over 240 million sextics, we did not find any sextics whose hessian curves had more than 11 ovals in  $\mathbb{RP}^2$ . These last two examples, which have 12 ovals in

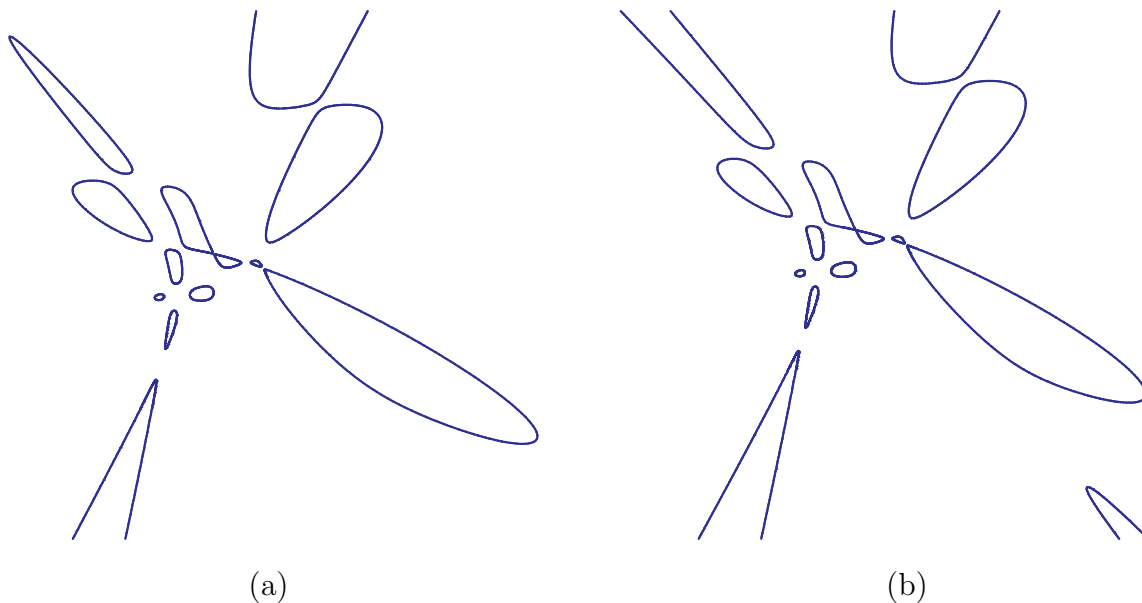


FIGURE 6. Hessians of sextics

$\mathbb{RP}^2$ , are perturbations of a sextic found in the search whose hessian curve had 11 ovals in  $\mathbb{RP}^2$ .

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INSTITUTO DE MATEMÁTICAS,, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, ÁREA DE LA INV. CIENTÍFICA, CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA, MÉXICO, D.F. 04510, MÉXICO

*E-mail address:* aortiz@matem.unam.mx

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA

*E-mail address:* sottle@math.tamu.edu

*URL:* <http://www.math.tamu.edu/~sottle>