APPENDICES FOR SCHUBERT POLYNOMIALS, THE BRUHAT ORDER, AND THE GEOMETRY OF FLAG MANIFOLDS

Abstract. These appendices are intended for informal distribution with the manuscript "Schubert polynomials, the Bruhat order, and the geometry of flag manifolds" and will not appear in the published version. They contain no results, only examples which we hope may illustrate some of the main results of that manuscript. Appendix A is intended to illustrate the geometric results, particularly of Section 5. We hope this may help others think about intersections of Schubert varieties. Appendix B is concerned with combinatorial and algebraic aspects of the manuscript. Many diagrams are enhanced with colour and may be viewed (in postscript) from either of the Authors' web pages.

Appendix A. Illustrating the geometric theorems

Throughout, let $e_1, \ldots, e_n$ be a fixed, ordered basis for the vector space $\mathbb{C}^n$. We use this basis to obtain a parameterization for Schubert cells and their intersections. Flags are represented by $n \times n$ matrices M: Let $(g_1, \ldots, g_n) := M \cdot e^T$ be the ordered basis given by the 'change of basis' matrix M. The ith row of M gives the components of g_i. Then M represents the flag $\langle\langle g_1, \ldots, g_n \rangle\rangle$. We adopt some conventions for the entries of M: a dot $(\bullet)$ will denote an entry of zero and an asterix $(*)$ an entry which may assume any value in $\mathbb{C}$. One last convention is that the flags $E_\bullet, F_\bullet$, etc. will always be defined to be $E_\bullet := \langle\langle e_1, \ldots, e_n \rangle\rangle$ and the 'primed' flags $E'_\bullet, F'_\bullet$, etc., which are opposite to their unprimed cousins, will be defined by $E'_\bullet := \langle\langle e_n, e_{n-1}, \ldots, e_2, e_1 \rangle\rangle$. We refer to these as the *standard flags*.

A.1. **Theorem E (*ii*).** In Theorem E (*ii*), we had $u \leq_k w$, $x \leq_k z$, and $wu^{-1} = zx^{-1}$ and we studied $X_{\omega_0 w} E_\bullet \bigcap X_u E'_\bullet$ and $X_{\omega_0 z} E_\bullet \bigcap X_x E'_\bullet$. The main result was that, in $Grass_k \mathbb{C}^n$,

$$\pi_k \left(X_{\omega_0 w} E_\bullet \bigcap X_u E'_\bullet \right) \;=\; \pi_k \left(X_{\omega_0 z} E_\bullet \bigcap X_x E'_\bullet \right).$$

The general case of Theorem E (*ii*) was reduced to Lemma 5.1.2, where w was Grassmannian of descent k, and $k < i \Longrightarrow u(i) = x(i)$ (and hence also $w(i) = z(i)$). The first example illustrates this case.

Let $n = 7$, $k = 4$, and

$$\begin{aligned} u &= 1436257 & \qquad x &= 4631257 \\ w &= 4567123 & \qquad z &= 5764123 \end{aligned}$$

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Note that $wu^{-1} = (1452)(367) = zx^{-1}$. The following matrices represent general flags in the Schubert cells $X^\circ_{\omega_0 w}E_\bullet$, $X^\circ_u E'_\bullet$, $X^\circ_{\omega_0 z}E_\bullet$, and $X^\circ_x E'_\bullet$, respectively:

$$\begin{matrix} * & * & * & 1 & \cdot & \cdot & \cdot \\ * & * & * & \cdot & 1 & \cdot & \cdot \\ * & * & * & \cdot & \cdot & 1 & \cdot \\ * & * & * & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad \begin{matrix} 1 & * & * & * & * & * & * \\ \cdot & \cdot & \cdot & 1 & * & * & * \\ \cdot & \cdot & 1 & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * \\ \cdot & 1 & \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix} \qquad \begin{matrix} * & * & * & * & 1 & \cdot & \cdot \\ * & * & * & * & \cdot & * & 1 \\ * & * & * & * & \cdot & 1 & \cdot \\ * & * & * & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad \begin{matrix} \cdot & \cdot & \cdot & 1 & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & 1 & \cdot & * & \cdot & * \\ 1 & * & \cdot & \cdot & * & \cdot & * \\ \cdot & 1 & \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$$

We chose w to be Grassmannian so that the cell $X^\circ_{\omega_0 w}E_\bullet$ has a particularly simple form. This gives an easy parameterization for the intersection of the two cells, $X^\circ_{\omega_0 w}E_\bullet \bigcap X^\circ_u E'_\bullet$. In the proof of Lemma 5.1.4 we describe how to find bases parameterized by $A := \{M \in M(w) \mid M \in X^\circ_u E_\bullet\}$. In practice, this method may be used to determine the subvariety A of $M(w)$.

First, let $g_1, \ldots, g_7$ be the rows of the following matrix, where $\alpha, \beta, \gamma, \delta, x, \rho, \sigma$, and τ are arbitrary elements of $\mathbb{C}$ with $\alpha\delta x\tau \neq 0$:

$$\begin{matrix} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & x & \rho & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

These parameters were chosen so that for each $j = 1, 2, 3, 4$, $g_j \in E_{w(j)} \bigcap E'_{n+1-u(j)}$ and does not lie in either of $E_{w(j)-1}$ or $E'_{n-u(j)}$, hence the 1's, the condition on α, δ, x, τ, and the 0's ($\cdot$) in their initial columns.

This matrix determines a flag $G_\bullet := \langle\langle g_1, \ldots, g_n \rangle\rangle \in X^\circ_{\omega_0 w}E_\bullet$, since it is in $M(w)$. Also, since $g_j \in E'_{n+1-u(j)} - E'_{n-u(j)}$ for $j \leq k$, at least $G_1, \ldots, G_k$ satisfy the conditions for the flag $G_\bullet$ to be in $X^\circ_u E'_\bullet$. The remaining conditions for $G_\bullet \in X^\circ_u E'_\bullet$,

$$G_{j-1} \bigcap E'_{n+1-u(j)} \subsetneq G_j \bigcap E'_{n+1-u(j)} \qquad \text{for} \qquad k < j,$$

impose additional restrictions on the parameters. In practice this means we seek conditions to ensure that $(\mathbb{C}^\times g_j + G_{j-1}) \bigcap E'_{n+1-u(j)}$ is non-empty. For instance, for $j = 6$, since $\langle g_5, g_5 \rangle = (*, *, 0, 0, 0, 0, 0)$ and $E_{n+1-u(6)} = (0, 0, 0, 0, *, *, *)$, some cancellation must occur. Indeed, since

$$\begin{aligned} -\alpha g_5 - \beta g_6 + g + 1 &= (0, 0, \gamma, 1, 0, 0, 0) \\ g_3 &= (0, 0, x, \rho, \sigma, 1, 0), \end{aligned}$$

we must have $\gamma\rho - x = 0$ in order that $(\mathbb{C}^\times g_6 + G_5) \bigcap E'_{n+1-u(6)} \neq \emptyset$. In the general situation, more complicated determinantal conditions may arise.

From these considerations, we arrive at two equivalent parameterizations for $X_{\omega_0 w}E_\bullet \bigcap X_u E'_\bullet$:

$$\begin{matrix} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad \begin{matrix} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ \cdot & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$$

For $\alpha, \ldots, \tau \in \mathbb{C}^\times$, both matrices represent the same flag in the intersection. To see this, let $g_1, \ldots, g_7$ be the basis determined by the first matrix, and $g'_1, \ldots, g'_7$ the basis determined by the second matrix. Then, by the definition of Schubert cells in §2.3, the flags $\langle\!\langle g_1, \ldots, g_7 \rangle\!\rangle \in X^\circ_{\omega_0 w} E_\bullet$ and $\langle\!\langle g'_1, \ldots, g'_7 \rangle\!\rangle \in X^\circ_u E'_\bullet$. Since $g_i = g'_i$ for $i = 1, 2, 3, 4$, and we have

$$\begin{aligned} g'_5 &= g_1 - \alpha g_5, \\ g'_6 &= g_3 - \rho(g_1 - \alpha g_5 - \beta g_6), \qquad \text{and} \\ g'_7 &= g_4 - \tau\left[g_3 - \rho(g_1 - \alpha g_5 - \beta g_6) - \sigma\left(g_2 - \delta(g_1 - \alpha g_5 - \beta g_6 - \gamma g_7)\right)\right], \end{aligned}$$

we see that $\langle\!\langle g_1, \ldots, g_7 \rangle\!\rangle = \langle\!\langle g'_1, \ldots, g'_7 \rangle\!\rangle$. Lastly, since $\ell(w) - \ell(u) = 12 - 5 = 7$, and $E_\bullet, E'_\bullet$ are opposite flags, we see that $X^\circ_u E'_\bullet \bigcap X^\circ_{\omega_0 w} E_\bullet$ is irreducible of dimension 7. Thus the matrices represent a 7-parameter family of flags in this intersection, which must be dense.

Similarly, (with the same restrictions on parameters), the two matrices below both represent the same flag in $X^\circ_{\omega_0 z} E_\bullet \bigcap X^\circ_x E'_\bullet$:

$$\begin{array}{ccccccc} \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

If $h_1, \ldots, h_7$ is the basis determined by the first matrix, then $h_1 = g_2$, $h_2 = g_4$, $h_3 = g_3$, and $h_4 = g_1$. Thus $\langle g_1, g_2, g_3, g_4 \rangle = \langle h_1, h_2, h_3, h_4 \rangle$, which proves

$$\pi_k\left(X_u E'_\bullet \bigcap X_{\omega_0 w} E_\bullet\right) \;=\; \pi_k\left(X_x E'_\bullet \bigcap X_{\omega_0 y} E_\bullet\right).$$

This is true even when u, w, x, z do not satisfy the extra hypotheses of Lemma 5.1.2. (One may construct a proof using the geometric analogs of the arguments that reduce Theorem E (*ii*) to Lemma 5.1.2.) We illustrate this with another example.

Here, let $n = 7$, $k = 3$, and

$$\begin{aligned} u &= 2134765 & \qquad x &= 2316475 \\ w &= 3571624 & \qquad z &= 3752164 \end{aligned}$$

Note that $wu^{-1} = (154)(2376) = zx^{-1}$. Then the following four matrices represent, respectively, the Schubert cells $X^\circ_{\omega_0 w} E_\bullet$, $X^\circ_u E'_\bullet$, $X^\circ_{\omega_0 z} E_\bullet$, and $X^\circ_x E'_\bullet$:

$$\begin{array}{ccccccc} * & * & 1 & \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & * & 1 & \cdot & \cdot \\ * & * & \cdot & * & \cdot & * & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & * & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} \cdot & 1 & * & * & * & * & * \\ 1 & \cdot & * & * & * & * & * \\ \cdot & \cdot & 1 & * & * & * & * \\ \cdot & \cdot & \cdot & 1 & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} * & * & 1 & \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & * & * & * & 1 \\ * & * & \cdot & * & 1 & \cdot & \cdot \\ * & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & * & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} \cdot & 1 & * & * & * & * & * \\ \cdot & \cdot & 1 & * & * & * & * \\ 1 & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & \cdot & 1 & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array}$$

Consider the (equivalent pairs of) parameterizations for flags in the intersections of the cells, $X^\circ_{\omega_0 w}E_\bullet \bigcap X^\circ_u E'_\bullet$ (the left-hand pair), and $X^\circ_{\omega_0 z}E_\bullet \bigcap X^\circ_x E'_\bullet$ (the right-hand pair):

$$
\begin{array}{ccccccc}
\cdot & \alpha & 1 & \cdot & \cdot & \cdot & \cdot \\
\beta & \alpha\gamma & \gamma & \delta & 1 & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & 1 \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & \cdot \\
\cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot & \alpha & 1 & \cdot & \cdot & \cdot & \cdot \\
\beta & \alpha\gamma & \gamma & \delta & 1 & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & 1 \\
\cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\
\cdot & \cdot & \cdot & \cdot & \sigma & \tau & 1
\end{array}
\qquad
\begin{array}{ccccccc}
\cdot & \alpha & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & 1 \\
\beta & \alpha\gamma & \gamma & \delta & 1 & \cdot & \cdot \\
\beta\sigma & \alpha\rho & \cdot & \cdot & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & \cdot \\
\beta & \alpha\gamma & \gamma & \delta & \cdot & \cdot & \cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot & \alpha & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \rho & \delta\sigma & \sigma & \tau & 1 \\
\beta & \alpha\gamma & \gamma & \delta & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\
\cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}
$$

To see that each pair of matrices does indeed give the same flag, let $g_1, \ldots, g_7$, $g'_1, \ldots, g'_7$, $h_1, \ldots, h_7$, and $h'_1, \ldots, h'_7$ be the bases given by the four matrices (read left-to-right). Then, for $i = 1, 2, 3$, $g_i = g'_i$ and $h_i = h'_i$. Also,

$$
\begin{aligned}
g'_4 &= -\alpha g_1 + g_2 - g_4 \\
g'_5 &= g_3 - g_5 \\
g'_6 &= g_3 - \sigma g'_4 - \rho(g_1 - g_6) \\
g'_7 &= g_3 - g_7
\end{aligned}
$$

and

$$
\begin{aligned}
h'_4 &= h_2 - \sigma h_3 + h_4 + (\gamma - \rho)h_1 \\
h'_5 &= h_3 - \gamma h_1 - h_5 \\
h'_6 &= h_2 - h_6 \\
h'_7 &= h_3 - h_7,
\end{aligned}
$$

thus, $\langle\langle g_1, \ldots, g_7 \rangle\rangle = \langle\langle g'_1, \ldots, g'_7 \rangle\rangle$ and $\langle\langle h_1, \ldots, h_7 \rangle\rangle = \langle\langle h'_1, \ldots, h'_7 \rangle\rangle$. As before, these parameterized bases give dense subsets of flags in each of $X_{\omega_0 w}E_\bullet \bigcap X_u E'_\bullet$ and $X_{\omega_0 z}E_\bullet \bigcap X_x E'_\bullet$. Moreover, since $\langle g_1, g_2, g_3 \rangle = \langle h_1, h_2, h_3 \rangle$, we see that

$$
\pi_k\left(X_{\omega_0 w}E_\bullet \bigcap X_u E'_\bullet\right) \;=\; \pi_k\left(X_{\omega_0 z}E_\bullet \bigcap X_x E'_\bullet\right).
$$

A.2. **Theorem G *(ii)*.** We complete Example 6.2.2, giving the geometric side of the story. The permutation (1978)(26354) is the disjoint product of $\zeta = (1978)$ and $\eta = (26354)$. Note that $u = 372186945 \leq_4 586913724 = (\zeta\eta)u =: w$. Let $G_\bullet$ and $G'_\bullet$ be the standard flags in $\mathbb{C}^9$. The following matrices parameterize the Schubert cells $X^\circ_{\omega_0 586913724}G_\bullet$ and $X^\circ_{372186945}G'_\bullet$:

$$
\begin{array}{ccccccccc}
\cdot & \cdot & 1 & * & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * & * \\
\cdot & 1 & \cdot & * & * & * & \cdot & * & * \\
1 & \cdot & \cdot & * & * & * & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & * & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccccccccc}
* & * & * & * & 1 & \cdot & \cdot & \cdot & \cdot \\
* & * & * & * & \cdot & * & * & 1 & \cdot \\
* & * & * & * & \cdot & 1 & \cdot & \cdot & \cdot \\
* & * & * & * & \cdot & \cdot & * & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & * & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & * & \cdot & * & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

As before, here are two parameterized matrices, each of which give bases defining the same flag in the intersection of the two cells:

$$\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -c & as & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & b & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

It is clearer to display two matrices 'on top of each other', with shading:

$$\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
1| & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & |\gamma & 1 \\
\cdot & -c & as| & \cdot & \cdot & |1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1| & \cdot| & 1 \\
\cdot & 1| & \cdot & |bs & s & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & b| & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

The vertical lines in the last 5 rows illustrate that the left- and right-sides of those rows come from different (equivalent) bases. The shading accentuates its 'block form': Let $Q = \{1,7,8,9\}$ and $P = \{2,4,5,7\} = u^{-1}(Q)$. The $P^c = \{1,3,6,8,9\} = u^{-1}(Q^c)$, where $Q^c = \{2,3,4,5,6\}$. Then the shaded regions are $(P \times Q) \bigcup (P^c \times Q^c)$. We see that $\zeta' := (1423)$ and $\eta' := (15243)$ are uniquely defined by $\phi_Q \zeta' = \zeta$ and $\phi_{Q^c} \eta' = \eta$. Moreover, we may define permutations v and w as in Lemma 5.2.1; let $v = 2134$ and $w = 21534$. Then

(1) $v \leq_2 \zeta' v = 3412$ and $w \leq_2 \eta' w = 45213$.
(2) $u = \varepsilon_{P,Q}(v,w)$ and $(\zeta\eta)u = \varepsilon_{P,Q}(\zeta' v, \eta' w)$.

For the last part of Lemma 5.2.1, let $F_\bullet, F'_\bullet$ be the standard flags in $\mathbb{C}^4$, and $E_\bullet, E'_\bullet$ the standard flags in $\mathbb{C}^5$. Then the following four matrices parameterize the Schubert cells $X^\circ_{\omega_0\zeta' v} F_\bullet$, $X^\circ_v F'_\bullet$, $X^\circ_{\omega_0 \eta' w} E_\bullet$, and $X^\circ_w E'_\bullet$:

$$\begin{array}{cccc}
* & * & 1 & \cdot \\
* & * & \cdot & 1 \\
1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot
\end{array}
\qquad
\begin{array}{cccc}
\cdot & 1 & * & * \\
1 & \cdot & * & * \\
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot
\end{array}
\qquad
\begin{array}{ccccc}
* & * & * & 1 & \cdot \\
* & * & * & \cdot & 1 \\
* & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccccc}
\cdot & 1 & * & * & * \\
1 & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & * & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot
\end{array}$$

Then the following two matrices parameterize the two intersections. Again, we have drawn two matrices on top of each other.

$$\begin{array}{cccc}
\cdot & \alpha & 1 & \cdot \\
\beta & \cdot & \gamma & 1 \\
1| & \cdot| & \gamma & 1 \\
\cdot & 1| & \cdot| & 1
\end{array}
\qquad
\begin{array}{ccccc}
\cdot & a & b & 1 & \cdot \\
c & \cdot & bs & s & 1 \\
-c & as| & \cdot & \cdot| & 1 \\
1| & \cdot & |bs & s & 1 \\
\cdot & a & b| & 1 & \cdot
\end{array}$$

Next, note that $G_\bullet = \psi_Q(E_\bullet, F_\bullet)$ and $G'_\bullet = \psi_{\omega_9 Q}(E'_\bullet, F'_\bullet)$. Finally, verifying that

$$\psi_P \left[\left(X_{\omega_4 \zeta' v} E_\bullet \bigcap X_v E'_\bullet \right) \times \left(X_{\omega_5 \eta' w} F_\bullet \bigcap X_w F'_\bullet \right) \right]$$

is equal to

$$X_{\omega_9(\zeta\eta)u}G_\bullet \bigcap X_u G'_\bullet .$$

may be done by comparing these parameterized matrices.

In the final part of the proof of Theorem G (*ii*), we compare images of these intersections under projections to Grassmannians. The row spans of the next three parameterized matrices represent $\pi_2\left(X^\circ_{\omega_4\zeta' v}E_\bullet \bigcap X^\circ_v E'_\bullet\right)$, $\pi_2\left(X^\circ_{\omega_4\eta' w}F_\bullet \bigcap X^\circ_w F'_\bullet\right)$, and $\pi_4\left(X^\circ_{\omega_9(\zeta\eta)u}G_\bullet \bigcap X^\circ_u G'_\bullet\right)$ in each of $Grass_2\mathbb{C}^4$, $Grass_2\mathbb{C}^5$, and $Grass_4\mathbb{C}^9$, respectively.

$$\begin{array}{cccc} \cdot & \alpha & 1 & \cdot \\ \beta & \cdot & \gamma & 1 \end{array} \qquad \begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bs & s & 1 \end{array} \qquad \begin{array}{ccccccccc} \cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\ \cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \end{array}$$

Thus

$$\pi_4\left(X_{\omega_9(\zeta\eta)u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{\omega_9 Q}(E'_\bullet, F'_\bullet)\right)$$

is equal to

$$\varphi_{2,2}\left(\pi_2\left(X_{\omega_4\zeta' v}E_\bullet \bigcap X_v E'_\bullet\right) \times \pi_2\left(X_{\omega_5\eta' w}F_\bullet \bigcap X_w F'_\bullet\right)\right).$$

Consider the images of $X_{\omega_4\zeta' v}E_\bullet \times X_{\omega_5\eta' w}F_\bullet$ and $X_v E'_\bullet \times X_w F'_\bullet$ under ψ_P in $\mathbb{F}\ell_9$:

$$\begin{array}{ccccccccc} \circ & * & * & * & 1 & \cdot & \cdot & \cdot & \cdot \\ * & \circ & \circ & \circ & \cdot & \circ & * & 1 & \cdot \\ \circ & * & * & * & \cdot & 1 & \cdot & \cdot & \cdot \\ * & \circ & \circ & \circ & \cdot & \cdot & * & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & 1 & \cdot & \cdot & \cdot & \circ & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccccc} \cdot & \cdot & 1 & * & * & * & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * & * \\ \cdot & 1 & \cdot & * & * & * & \cdot & \circ & \circ \\ 1 & \cdot & \cdot & \circ & \circ & \circ & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \circ & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array}$$

This should be compared with the first figure of this section, which shows the cells $X^\circ_{\omega_0 586913724}G_\bullet$ and $X^\circ_{372186945}G'_\bullet$. Here, the circles ($\circ$) indicate the 'surprise' entries of 0; those which are not zero in that first figure. This illustrates the two inclusions, and serves to illustrate Lemma 4.5.1:

$$\begin{array}{rcl} \psi_P\left(X_{\omega_4\zeta' v}E_\bullet \times X_{\omega_5\eta' w}F_\bullet\right) & \subset & X^\circ_{\omega_0 586913724}G_\bullet \\ \psi_P\left(X_v E'_\bullet \times X_w F'_\bullet\right) & \subset & X^\circ_{372186945}G'_\bullet \end{array}$$

A.3. **Theorem H.** We illustrate the 'cyclic shift'. Let $u = 21354$ and $w = 45123$ so that $wu^{-1} = \zeta = (15243)$. Define $x, z \in \mathcal{S}_5$ as in the proof of Theorem H′ (§ 5.3) to be $x = 31245$ and $z = 53124$. Then $zx^{-1} = (13542) = \zeta^{(12345)}$. Here, $p = 4$, $m = 1$, and $l = 2$. Let $F_\bullet, F'_\bullet$ be the standard flags for $\mathbb{C}^5$. We define $G_\bullet = \langle\langle e_5, e_1, e_2, e_3, e_4\rangle\rangle$ and $G'_\bullet = \langle\langle e_4, e_3, e_2, e_1, e_5\rangle\rangle$. Then, with respect to these flags, the Schubert cells $X^\circ_{\omega_0 w}F_\bullet$, $X^\circ_u F'_\bullet$, $X^\circ_{\omega_0 z}G_\bullet$, and $X^\circ_x G'_\bullet$ are:

$$\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ * & * & * & 1 & \cdot \\ * & * & * & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{array} \quad \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \cdot & 1 & * & * & * \\ 1 & \cdot & * & * & * \\ \cdot & \cdot & 1 & * & * \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \qquad \begin{array}{ccccc} e_5 & e_1 & e_2 & e_3 & e_4 \\ * & * & * & * & 1 \\ * & * & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \quad \begin{array}{ccccc} e_5 & e_1 & e_2 & e_3 & e_4 \\ \cdot & \cdot & 1 & * & * \\ 1 & * & \cdot & * & * \\ \cdot & 1 & \cdot & * & * \\ \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

Here, since the flags are different, the columns of the matrices on the right correspond to different elements of our fixed basis, e_1, e_2, e_3, e_4, e_5, as indicated.

Here are two matrices giving (equivalent) parameterized bases for flags in the intersection of the cells $X^\circ_{\omega_0 w}F_\bullet \bigcap X^\circ_u F'_\bullet$:

$$\begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bd & d & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bd & d & 1 \\ \cdot & \cdot & bd & d & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array}$$

Here $a, b, c, d \in \mathbb{C}^\times$, showing that $(\mathbb{C}^\times)^4$ parameterizes the set A of the intersection of cells. Let $g_1, \ldots, g_5$ be the basis given by the left matrix and $g'_1, \ldots, g'_5$ the basis given by the right matrix. Since

$$g_2(a,b,c,d) \;=\; e_5 \;+ c\, e_1 + bd\, e_3 + d\, e_4,$$

$(\beta_1, \beta_2, \beta_3, \beta_4) = (c, 0, bd, d)$ are regular functions on A. Also, since

$$e_5 = -d\, g_1 + g_2 - c\, g_3 + da\, g_4,$$

$\delta_1 = -d$, $\delta_3 = -c$, and $\delta_4 = da$ are regular functions on A with δ_4 nowhere vanishing. The bases $h_1, \ldots, h_5$ and $h'_1, \ldots, h'_5$ defined in the proof of Theorem H$'$, are parameterized by the following two matrices:

$$\begin{array}{ccccc} \cdot & \cdot & a & b & 1 \\ 1 & -c & da & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & \cdot & a & b & 1 \\ 1 & -c & da & \cdot & \cdot \\ \cdot & c & \cdot & bd & d \\ \cdot & \cdot & \cdot & bd & d \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

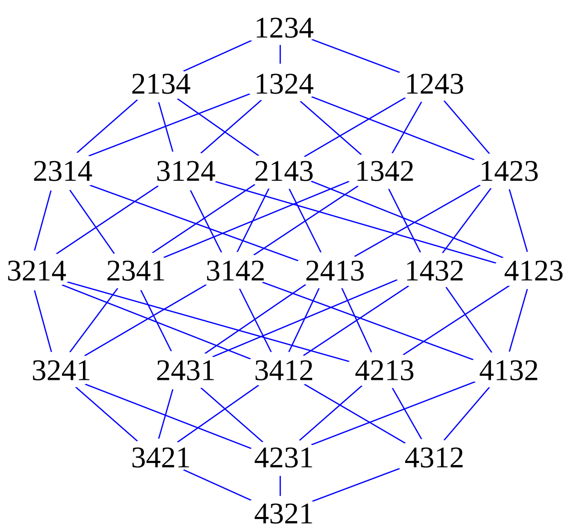
These matrices give equivalent bases, and hence define the same flag. Also, comparing them to the rightmost two matrices in the first figure of this subsection, shows this flag is in the intersection $X^\circ_{\omega_0 z}G_\bullet \bigcap X^\circ_x G'_\bullet$. Since the first two rows of each matrix have the same span,

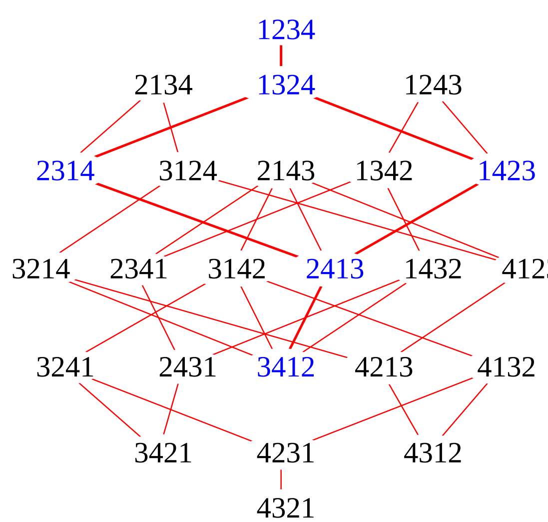
$$\pi_2\left(X^\circ_{\omega_0 w}F_\bullet \bigcap X^\circ_u F'_\bullet\right) \;=\; \pi_2\left(X^\circ_{\omega_0 z}G_\bullet \bigcap X^\circ_x G'_\bullet\right),$$

which is the main geometric result needed to deduce Theorem H$'$.

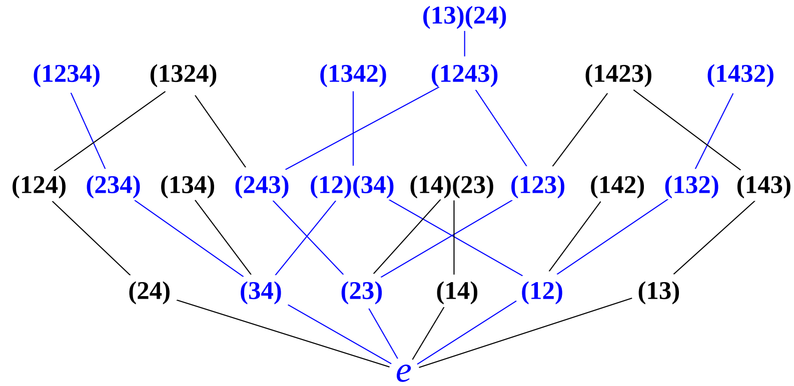
Appendix B. Combinatorial and algebraic examples

B.1. **Suborders of $\mathcal{S}_4$.** The Bruhat order is one of our main objects of study in this paper. Here is a picture of the (full) Bruhat order and the 2-Bruhat order on $\mathcal{S}_4$.

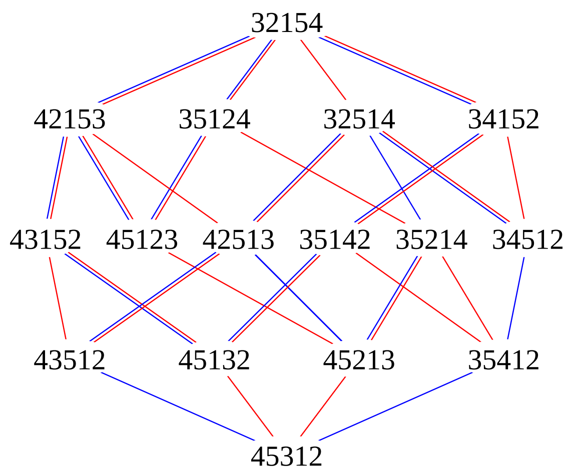




For comparison, here is the $\preceq$-order on $\mathcal{S}_4$ (reproduced from §3.2).



B.2. **Chains in the P-Bruhat order.** Theorem B describes the relation between chains in the P-Bruhat order and the structure constants $c^w_{u\,v}$, when v is a minimal coset representative in vP. We consider an instance of this. Let $P := \langle (1,2),(4,5)\rangle \subset \mathcal{S}_5$. Then $32154 \leq_P 45312$ and this is the interval $[32154,\ 45312]_P$:



The multiple edges are those with two possible colourings. One may verify that $f^{45312}_{32154}(P) = 57$. To check Theorem B, we first compute $c^{45312}_{32154\ v}$ for those $v \in \mathcal{S}_5$ of length 4 which are minimal in their P-coset.

$$25134, \quad 34125, \quad 24315, \quad 15324, \quad 14523, \quad \text{and} \quad 23514.$$

The first two are Grassmannian of descent 2, and the last two are Grassmannian of descent 3. Since $32154 \not\leq_2 45312$, we have

$$c^{45312}_{32154\ 25134} \quad = \quad c^{45312}_{32154\ 34125} \quad = \quad 0,$$

Let $\zeta = (13425)$. Then $45312 = \zeta \cdot 32145$ and $(13425)^{(12345)} = (12435)$. Since $(12435) = v(\boxplus, 3) \cdot v(\square, 3)^{-1}$ and $32154 \leq_3 45312$, Theorem H implies

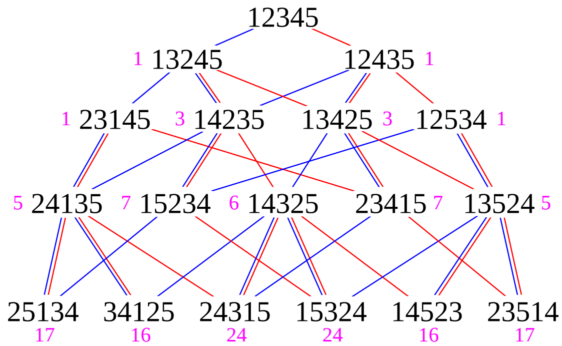
$$c^{45312}_{32154\ 14523} \quad = \quad c^{\boxplus}_{\boxplus} \quad = \quad 1 \qquad \text{and} \qquad c^{45312}_{32154\ 23514} \quad = \quad c^{\boxplus}_{\boxplus} \quad = \quad 1.$$

Next, let $F_\bullet, F'_\bullet, F''_\bullet$ be in general position. If $E_\bullet \in X_{15324}F_\bullet \bigcap X_{32154}F'_\bullet$, then $E_2 \subset F'_4$ and $E_2 \supset F_1$, contradicting $F_\bullet$ and $F'_\bullet$ in general position. Thus

$$c^{45312}_{32154\ 15324} \quad = \quad \#\left(X_{15324}F_\bullet \bigcap X_{32154}F'_\bullet \bigcap X_{\omega_0 45312}F''_\bullet\right) \quad = \quad 0.$$

To compute $c^{45312}_{32154\ 24315} = \deg\left(\mathfrak{S}_{\omega_0 45312} \cdot \mathfrak{S}_{32154} \cdot \mathfrak{S}_{24315}\right)$, note that $\mathfrak{S}_{\omega_0\,45312} = \mathfrak{S}_{21354} = \mathfrak{S}_{(1,2)} \cdot \mathfrak{S}_{(4,5)}$. Two applications of Monk's formula show $c^{45312}_{32154\ 24315} = 1$. (The other computations could also have proceeded via Monk's formula.)

To compute $f_e^v(P)$ for these minimal coset representatives, consider the part of the P-Bruhat order rooted at e and restricted to permutations of length at most 4:



The small numbers adjacent to each permutation v are $f_e^v(P)$. Thus

$$\sum_v f_e^v(P)\, c_{32154\, v}^{45312} \quad = \quad 17\cdot 0 \;+\; 16\cdot 0 \;+\; 24\cdot 1 \;+\; 24\cdot 0 \;+\; 16\cdot 1 \;+\; 17\cdot 1 \quad = \quad 57,$$

which equals $f_{32154}^{45312}(P)$.

B.3. **Instance of Theorem D.** We consider $\Psi_{\{1,3,5,\ldots\}}(\mathfrak{S}_{516432})$.

$$\begin{aligned}\mathfrak{S}_{516432} \;=\; & x_1^4x_2^2x_3^3x_5 + x_1^4x_2x_3^3x_4x_5 + x_1^4x_3^3x_4^2x_5\\ & + x_1^4x_2^3x_3^2x_5 + x_1^4x_2^2x_3^3x_4 + x_1^4x_2^2x_3^2x_4x_5 + x_1^4x_2x_3^3x_4^2 + x_1^4x_2x_3^2x_4^2x_5\\ & + x_1^4x_2^3x_3^2x_4 + x_1^4x_2^3x_3x_4x_5 + x_1^4x_2^2x_3^2x_4^2 + x_1^4x_2^2x_3x_4^2x_5\\ & + x_1^4x_2^3x_3x_4^2 + x_1^4x_2^3x_4^2x_5.\end{aligned}$$

$\Psi_{\{1,3,5,\ldots\}}(\mathfrak{S}_{516432}) = \mathfrak{S}_{516432}(y_1, z_1, y_2, z_2, y_3, z_3, \ldots)$, which is

$$\begin{aligned} & y_1^4y_2^3y_3(z_1^2 + z_1z_2 + z_2^2) \;+\; y_1^4y_2^2y_3(z_1^3 + z_1z_2^2 + z_1^2z_2) \;+\; y_1^4y_2^3(z_1^2z_2 + z_1z_2^2)\\ & \qquad + \;(y_1^4y_2^2 + y_1^4y_2y_3)(z_1^3z_2 + z_1^2z_2^2) \;+\; (y_1^4y_2 + y_1^4y_3)z_1^3z_2^2.\end{aligned}$$

Using the definition of Schubert polynomials in §2.2, one may check

$$\begin{array}{lllllll} \mathfrak{S}_{54213} & = & x_1^4x_2^3x_3 & \qquad & \mathfrak{S}_{53214} & = & x_1^4x_2^2x_3\\ \mathfrak{S}_{54213} & = & x_1^4x_2^3 & & \mathfrak{S}_{53124} & = & x_1^4x_2^2\\ \mathfrak{S}_{52314} & = & x_1^4x_2x_3 & & \mathfrak{S}_{51324} & = & x_1^4x_2 + x_1^4x_3\end{array}$$

The Schubert polynomials $\mathfrak{S}_w$ for $w \in \mathcal{S}_4$ are indicated in Figure 1. The Schubert polynomial $\mathfrak{S}_w$ is written below the permutation w, and these data are displayed at the vertices of the permutahedron (Cayley graph of $\mathcal{S}_4$). The divided difference operators are displayed on the edges of this figure.

We see that $\Psi_{\{1,3,5,\ldots\}}\mathfrak{S}_{516432} = \mathfrak{S}_{516432}(y_1, z_1, y_2, z_2, y_3, z_3, \ldots)$ is equal to

$$\begin{aligned} & \mathfrak{S}_{54213}(y)\mathfrak{S}_{1423}(z) \;+\; \mathfrak{S}_{53214}(y)\left[\mathfrak{S}_{4123}(z) + \mathfrak{S}_{2413}(z)\right] \;+\; \mathfrak{S}_{54123}(y)\mathfrak{S}_{2413}(z)\\ & \quad + \;\left[\mathfrak{S}_{53124}(y) + \mathfrak{S}_{52314}(y)\right]\left[\mathfrak{S}_{4213}(z) + \mathfrak{S}_{3412}(z)\right] \;+\; \mathfrak{S}_{51324}(y)\mathfrak{S}_{4312}(z).\end{aligned}$$

B.4. **Automorphisms of $(\mathcal{S}_\infty, \preceq)$.** The definition of the k-Bruhat orders imply that if $u, w \in \mathcal{S}_n$, and $k < n$, then the following are equivalent:

$$u \leq_k w \qquad \omega_0 w \leq_k \omega_0 u \qquad w\omega_0 \leq_{n-k} u\omega_0 \qquad \omega_0 u \omega_0 \leq_{n-k} \omega_0 w \omega_0.$$

These induce the following isomorphisms (which were stated in Theorem 3.2.3) of intervals in the $\preceq$-order on $\mathcal{S}_\infty$. Suppose $\zeta \in \mathcal{S}_n$ and $\overline{\zeta} = \omega_0\zeta\omega_0$. Then

$$[e, \zeta]_\preceq \quad \simeq \quad [e, \overline{\zeta}^{-1}]_\preceq^{\mathrm{op}} \quad \simeq \quad [e, \zeta^{-1}]_\preceq^{\mathrm{op}} \quad \simeq \quad [e, \overline{\zeta}]_\preceq.$$

These are illustrated in the posets displayed in Figures 2 and 3.

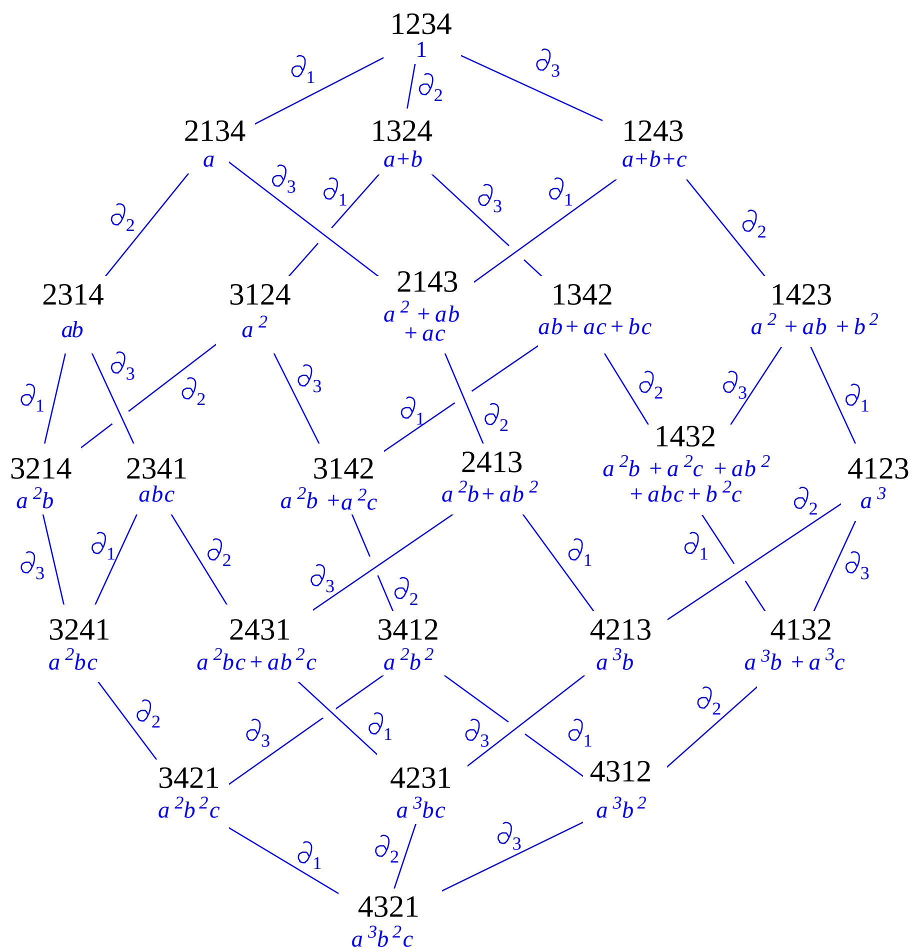


FIGURE 1. Schubert polynomials in $\mathcal{S}_4$

B.5. **Canonical algorithms?** Besides Algorithm 3.1.1, there are three other 'canonical' algorithms for finding a chain between u and w when $u \leq_k w$, each induced from Algorithm 3.1.1 by one of the automorphisms of the previous section. For example, here is one.

Algorithm B.5.1 (Produces a chain in the k-Bruhat order)**.**
`input:` *Permutations* $u, w \in \mathcal{S}_\infty$ *with* $u \leq_k w$.
`output:` *A chain in the* k*-Bruhat order from* w *to* u.
Output w. *While* $u \neq w$, *do*

1 *Choose* $a \leq k$ *with* $w(a)$ *maximal subject to* $u(a) < w(a)$.
2 *Choose* $k < b$ *with* $w(b)$ *minimal subject to* $w(b) \leq u(a) < u(b)$.
3 $u := u(a,b)$, *output* u.

In general, these algorithms produce different chains. In S_7, consider the two permutations $2317546 <_3 4671235$. Here are chains produced by the four algorithms:

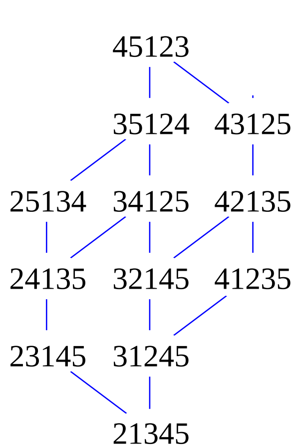
2317546	2317546	2317546	2317546
2417536	2417536	2371546	2371546
2517436	2517436	2571346	2571346
2617435	4517236	2671345	3571246
4617235	4617235	3671245	4571236
4671235	4671235	4671235	4671235

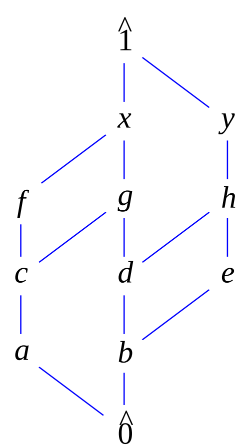
Here are the four algorithms for producing chains in $[e,\zeta]_{\preceq}$:

Algorithm B.5.2 (Chains in $\prec$-order). `input:` *A permutations* $\zeta \in \mathcal{S}_\infty$.
`output:` *Chains in* $[e,\zeta]_{\preceq}$.

I *Output* ζ. *While* $\zeta \neq e$, *do*
  1 *Choose* α *minimal such that* $\alpha < \zeta(\alpha)$.
  2 *Choose* β *maximal with* $\zeta(\beta) < \zeta(\alpha) \leq \beta$.
  3 $\zeta := \zeta(\alpha,\, \beta)$, *output* ζ.
II *Output* ζ. *While* $\zeta \neq e$, *do*
  1 *Choose* β *maximal such that* $\beta > \zeta(\beta)$.
  2 *Choose* α *minimal with* $\zeta(\alpha) > \zeta(\beta) \geq \alpha$.
  3 $\zeta := \zeta(\alpha,\, \beta)$, *output* ζ.
III *Output* e. *While* $\zeta \neq e$, *do*
  1 *Choose* $\zeta(\alpha)$ *maximal such that* $\alpha < \zeta(\alpha)$.
  2 *Choose* $\zeta(\beta)$ *minimal with* $\zeta(\beta) \leq \alpha < \beta$.
  3 $\zeta := \zeta(\alpha,\, \beta)$, *output* $(\alpha,\, \beta)$.
IV *Output* e. *While* $\zeta \neq e$, *do*
  1 *Choose* $\zeta(\beta)$ *minimal such that* $\beta > \zeta(\beta)$.
  2 *Choose* $\zeta(\alpha)$ *maximal with* $\zeta(\alpha) \geq \beta > \alpha$.
  3 $\zeta := \zeta(\alpha,\, \beta)$, *output* $(\alpha,\, \beta)$.

B.6. **Simplicial complexes and $\leq_k$.** In the theory of partially ordered sets, one often constructs a simplicial complex $\Delta(P)$ from a poset, P. We compute one such for an interval in the k-Bruhat order, which shows these intervals are not in general shellable. We illustrate this with one example drawn from this paper. In Example 3.2.4, we considered the interval $[21342,\ 45123]_2$. We display that interval below, together with the Hasse diagram of an isomorphic poset:

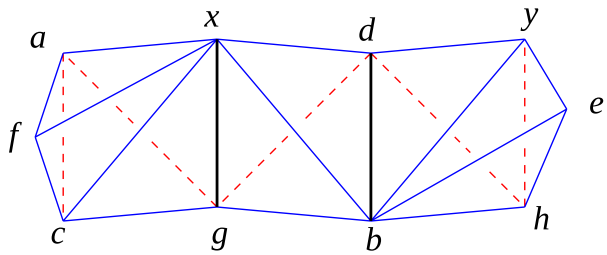




The simplicial complex $\Delta(P)$ associated to a poset P has as simplices all chains, including the non-maximal ones. In our case above, the maximal simplices are

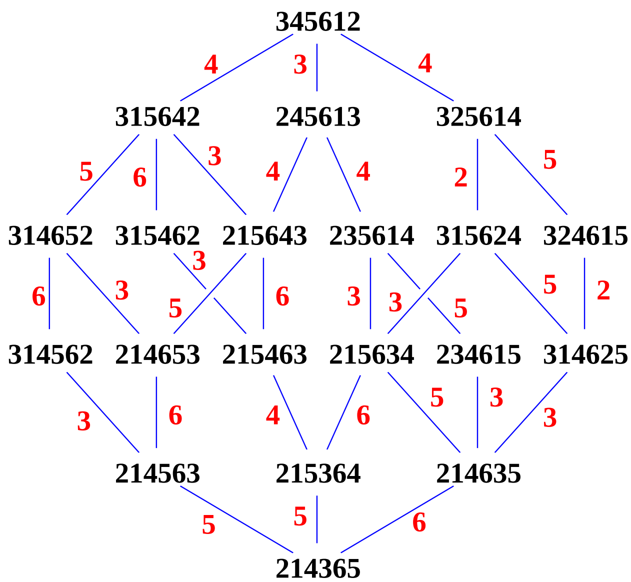
$$\{a,c,f,x\},\ \{a,c,g,x\},\ \{b,d,g,x\},\ \{b,d,h,y\},\ \{b,e,h,y\}.$$

While $(\{a,c,f,x\},\{a,c,g,x\})$ and $(\{b,d,h,y\},\{b,e,h,y\})$ are attached along facets ($\{a,c,x\}$ and $\{b,h,y\}$, respectively), the pairs $(\{a,c,g,x\},\{b,d,g,x\})$ and $(\{b,d,g,x\},\{b,d,h,y\})$ are not. They are attached along codimension 2 faces, $\{g,x\}$ and $\{b,d\}$, respectively. Thus this simplicial complex is not shelable. Below, we display a geometric realization of this simplicial complex:



B.7. **Schensted insertion and the $c^w_{u\,v(\lambda,k)}$.** In §6.3, we discussed how the conclusion of Theorem F′ holds for many permutations in $\mathcal{S}_6$, even most which are not skew permutations. We illustrate that here.

Let $\zeta = (145236)$. Then $214365 \leq_4 \zeta \cdot 214365 = 345612$. In Figure 2, we display the labeled Hasse diagram of $[214365,\ 345612]_4$ and beside it a table of the words of the 14 chains in this interval, each displayed above its insertion and recording tableau.

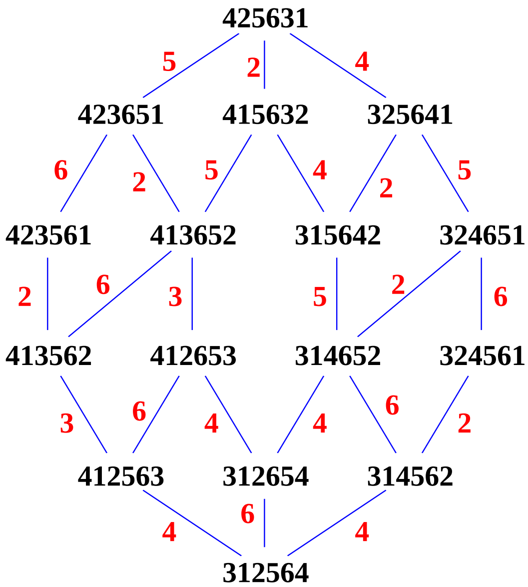


56543	65324	63543	65343	
6 5 5 4 4 3 35 12	6 4 5 3 3 2 24 15	6 5 5 4 4 2 33 13	6 5 5 3 4 2 33 14	
56343	56324	54643	54634	54364
5 5 46 34 33 12	5 4 36 35 24 12	5 5 46 24 34 13	5 4 46 25 34 13	5 3 46 25 34 14
56354	56534	53654	63524	63254
6 5 55 34 34 12	6 4 55 35 34 12	6 5 55 24 34 13	6 4 35 25 24 13	6 3 35 25 24 14

FIGURE 2. Labeled Hasse diagram of $[214365,\ 345612]_4$ and Schensted insertion

Note that $\eta := (125634) = \zeta^{(123456)}$ and $312564 \leq_4 \eta \cdot 312564 = 425631$. We continue this example, and illustrate Theorem H. In Figure 3 are the labeled Hasse diagram of $[312564,\ 425631]_4$, and the insertion and recording tableaux for all 14 chains in this interval.

For these last two intervals, it is interesting to view them with the permutation $v \in [u, w]_k$ replaced by the geometric graph of vu^{-1}, as illustrated in Figure 4. This gives an idea of the effect of a 'cyclic shift' on the $\preceq$-order.



46542	64325	64542	64352
6 5	6 4	6 5	6 5
5 4	4 3	4 4	4 3
3 3	3 2	3 2	3 2
24 12	25 15	25 13	25 14

46254	46524	42654	64524	64254
6 5	6 4	6 5	6 4	6 3
45 34	45 35	45 24	45 25	45 25
24 12	24 12	24 13	24 13	24 14

46352	46325	43652	43625	43265
4 5	4 4	4 5	4 4	4 3
36 34	36 35	36 24	36 25	36 25
25 12	25 12	25 13	25 13	25 14

Figure 3. Labeled Hasse diagram of $[312564, 425631]_4$ and Schensted insertion

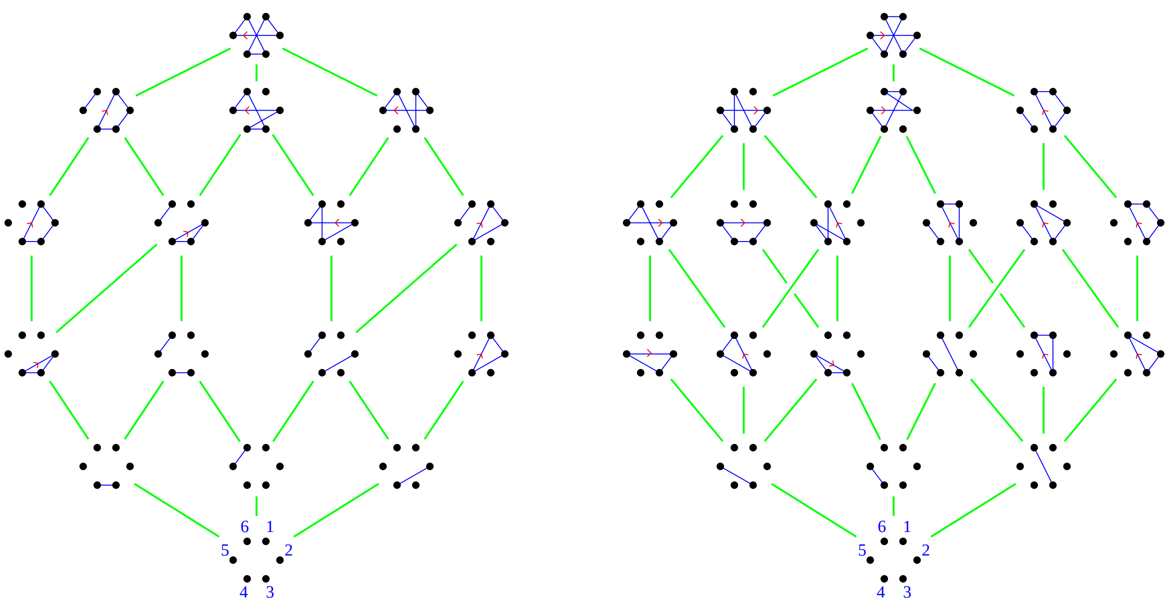
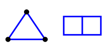


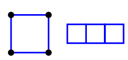
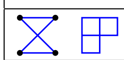
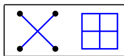
Figure 4. Geometric graphs of permutations in $[e, \zeta]_{\preceq}$ and $[e, \eta]_{\preceq}$

B.8. **Irreducible derangements, geometric graphs, and cyclic shift.** Here, we give tables displaying derangements in small symmetric groups that are irreducible. This is a companion to Section 6. The skew permutations are in **boldface**. They are grouped together under the equivalence relation generated by cyclic shift, inversion, and conjugation by the longest element. The permutations in a row are those in a single orbit under 'cyclic shift', which is conjugation by the long cycle $(1\,2\,\ldots\,n)$. We display the geometric graph only for one permutation in an equivalence class. Lastly, we also display a skew shape κ for which $c^{\zeta}_{\lambda} = c^{\kappa}_{\lambda}$ for λ fitting in a box of size $k \times (n-k)$.

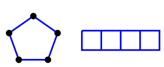
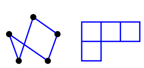
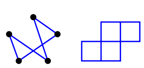
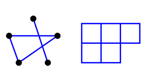
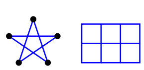
$\mathcal{S}_3$:

shape	k	permutations
	1	**(132)**
	2	**(123)**

$\mathcal{S}_4$:

shape	k	permutations
	1	**(1432)**
	3	**(1234)**
	2	**(1243)** (1423) **(1342)** (1324)
	2	**(13)(24)**

$\mathcal{S}_5$:

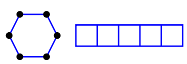
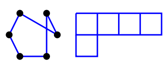
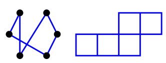
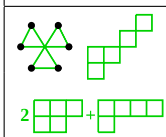
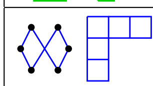
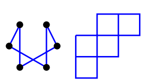
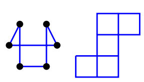
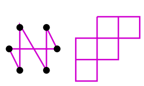
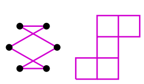
shape	k	permutations
	1	**(15432)**
	4	**(12345)**
	2	**(12543)** (15423) (15342) **(14532)** (14325)
	3	**(35421)** (13245) (12435) **(12354)** (15234)
	2	**(13542)** (15324) (14352) (13254) (15243)
	3	**(12453)** (14235) (12534) (14523) (13425)
	2	**(13)(254)** (24)(153) **(35)(142)** (14)(253) (25)(143)
	3	**(13)(245)** (24)(135) **(35)(124)** (14)(235) (25)(134)
	2	**(14253)**
	3	**(13524)**

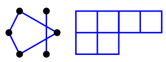
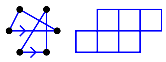
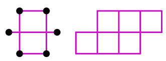
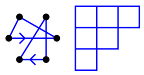
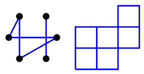
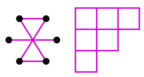
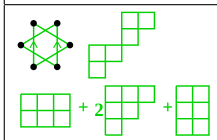
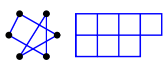
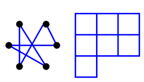
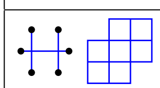
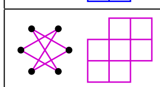
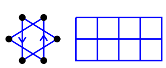
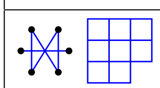
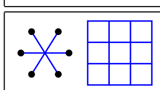
In $\mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ all permutations are equivalent to a skew shape, but in $\mathcal{S}_6$, the situation is different. Here, the graphs and shapes which are equivalent to skew shapes are displayed in blue, those for which we know the shape by path-counting and geometry/algebra are in maroon, and in green, we display the 6 for which the restriction on κ (namely $\kappa \subset (n-k)^k$) is necessary. These are

$$(125634),\ (145236),\ (143652),\ (163254),\ (153)(246),\ \text{and}\ (135)(264).$$

We also first list the minimal permutations, those for which $|\zeta| = 5$.

Minimal Permutations in $\boldsymbol{\mathcal{S}_6}$:

shape	k	permutations
	1	**(165432)**
	5	**(123456)**
	2	**(126543)** (165423) (165342) (164532) **(156432)** (154326)
	4	**(345621)** (324561) (243561) (235461) **(234651)** (623451)
	2	**(146532)** (164325) (154362) (132654) (165243) (163542)
	4	**(235641)** (523461) (263451) (456231) (342561) (245361)
	2	**(136542)** (142356) (125346) (123645) (156234) (134526)
	4	**(245631)** (653241) (643521) (546321) (432651) (625341)
	2	(143652) (163254)
	4	(256341) (452361)
	3	**(123654)** (165234) (163452) **(145632)** (143256) (125436)
	3	**(124653)** (164235) (153462) (132645) (156243) (135426)
	3	**(356421)** (532461) (264351) (546231) (342651) (624531)
	3	**(134652)** (163245) (143562) (132546) (124365) (162354)
	3	**(256431)** (542361) (265341) (645231) (563421) (453261)
	3	(125463) (142365) (162534) (136452) (156324) (143526)
	3	(132564) (152436) (126354) (146523) (163425) (145362)
	3	(153426) (126453) (156423)
	3	(624351) (354621) (324651)

shape	k	permutations
	2	**(13)(2654)** (24)(1653) (35)(1642) **(46)(1532)** (15)(2643) (26)(1543)
	4	**(13)(4562)** (24)(3561) (35)(2461) **(46)(2351)** (15)(3462) (26)(3451)
	2	**(142)(365)** (164)(253) (152)(364) (154)(263) (143)(265) (163)(254)
	4	**(241)(563)** (461)(352) (251)(463) (451)(362) (341)(562) (361)(452)
	2	(25)(1643) (36)(1542) (14)(2653)
	4	(25)(1346) (36)(1245) (14)(2356)
	3	**(124)(365)** (164)(235) (152)(346) (145)(263) (143)(256) (136)(254)
	3	**(421)(564)** (461)(532) (251)(643) (541)(362) (341)(652) (631)(452)
	3	**(13)(2564)** (24)(1536) (35)(1264) (46)(1523) (15)(2634) (26)(1453)
	3	**(13)(2465)** (24)(6351) (35)(4621) (46)(3251) (15)(4362) (26)(3541)
	3	**(46)(1253)** (15)(2364) (26)(1534) (13)(2645) (24)(1563) (35)(1426)
	3	**(46)(3521)** (15)(4632) (26)(4351) (13)(5462) (24)(3651) (35)(6241)
	3	(25)(1436) (36)(1254) (14)(2365) (25)(1634) (36)(1452) (14)(2563)
	3	(153)(246) (135)(264)
	2	**(142653)** (164253) **(153642)** (153264) (152643) (154263)
	4	**(356241)** (352641) **(246351)** (462351) (346251) (362451)
	3	**(125364)** (152364) (152634) (145263) **(142563)** (142536)
	3	**(463521)** (463251) (436251) (362541) **(365241)** (635241)
	3	**(13)(25)(46)** (15)(24)(36) (14)(26)(35)
	3	(142635) (146253) (136425) (153624) (135264) (152463)
	2	**(153)(264)**
	4	**(135)(246)**
	3	**(25)(1364)** (36)(1524) (14)(2635) **(25)(1463)** (36)(1425) (14)(2536)
	3	**(14)(25)(36)**

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E-mail address, Frank Sottile: `sottile@math.toronto.edu`
URL, Frank Sottile: `http://www.math.toronto.edu/~sottile`