Real Enumerative Geometry and Effective Algebraic Equivalence [2]

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We study when a problem in enumerative geometry may have all of its solutions be real and show that many Schubert-type enumerative problems on some flag manifolds can have all of their solutions be real. Our particular focus is how to use the knowledge that one problem can have all its solutions be real to deduce that other, related problems do as well. The primary technique is to study deformations of intersections of subvarieties into simple cycles. These methods may also be used to give lower bounds on the number of real solutions that are possible for a given enumerative problem.

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1 Introduction

Determining the common zeroes of (overdetermined) systems of polynomials is further complicated over non-algebraically closed fields such as the real numbers. We consider a variant of this problem, that of 'solving' a system of subvarieties of an algebraic variety, in other words, problems of enumerative geometry. We seek real solutions to such problems; in particular we ask when a given problem of enumerative geometry can have all its solutions be real. We call such a problem *fully real.*

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Little is known about enumerative geometry from this perspective. Since the Bézout bound may be attained for real polynomials, the problem of intersecting hypersurfaces in projective space is fully real. Khovanskii [13] considers intersecting hypersurfaces in a torus defined by few monomials and shows the number of real zeros are at most a fraction of the number of complex zeroes. Fulton, and more recently, Ronga, Tognoli and Vust [18] have shown the problem of 3264 plane conics tangent to five given conics is fully real. The author [21] has shown all problems of enumerating lines incident on linear subspaces of projective space are fully real.

We ask: How can the knowledge that one enumerative problem is fully real be used to infer that a related problem is fully real? We give several procedures to accomplish this inference and examples of their application, extending the list of enumerative problems known to be fully real.

This approach involves studying intersections of any dimension, not just the zero-dimensional intersections of enumerative problems. Our technique is to deform general intersection cycles into unions of simpler cycles. This modification of the classical method of degeneration was suggested by Chiavacci and Escamilla-Castillo [5], who investigated these questions for Grassmannians. If such deformations are described explicitly for an enumerative problem, (as they are in [21,22]) then it may be possible to use homotopy continuation methods [1] to find approximate (real) solutions for that problem. This has been accomplished for hypersurfaces in a *complex* torus [6,12].

Let $\alpha_1, \ldots, \alpha_a$ be cycle classes spanning the Chow ring of a smooth variety X. For cycle classes $\beta_1, \ldots, \beta_b$, there exist integers c_i for $i = 1, \ldots, a$ such that

$$\prod_{i=1}^{b} \beta_i \;=\; \sum_{i=1}^{a} c_i \cdot \alpha_i, \tag{1}$$

in the Chow ring of X. When the c_i are nonnegative, the product formula (1) has a geometric interpretation: For each $i = 1, \ldots, b$, let Y_i be a subvariety with cycle class β_i, and suppose that $Y_1, \ldots, Y_b$ meet generically transversally in a cycle Y. Then (1) asserts that Y is algebraically equivalent to a cycle $Z := Z_1 \cup \cdots \cup Z_a$, where Z_i has c_i components, each with cycle class α_i. That is, (roughly) there exists an algebraic deformation of Y into Z. This algebraic equivalence is *effective* if one may deform the intersection cycle Y into the cycle Z through cycles which are themselves intersections of the same form as Y. Thus effective algebraic equivalence is a concrete geometric manifestation of identities (1) in the Chow ring of X. If the cycles $Y_1, \ldots, Y_b$ and each component of Z are defined over $\mathbb{R}$, and the intermediate cycles are also intersections of real subvarieties, then the effective algebraic equivalence is *real*.

Real effective algebraic equivalence can be used to show an enumerative problem is fully real, or more generally, to obtain lower bounds on the maximal number of real solutions: In the situation of the previous paragraph, suppose the cycles $Y_1, \ldots, Y_b, W_1, \ldots, W_c$ intersect transversally in a zero-cycle. Further suppose that $Z \bigcap W_1 \bigcap \cdots \bigcap W_c$ is transverse, zero-dimensional, and has at least d real points. We had that Z is a deformation of intersections of the form $Y_1' \bigcap \cdots \bigcap Y_b'$, where each Y_i' is real and has cycle class β_i. Thus there exist real cycles $Y_1', \ldots, Y_b'$ such that $Y_1' \bigcap \cdots \bigcap Y_b' \bigcap W_1 \bigcap \cdots \bigcap W_c$ is transverse, zero-dimensional, and has at least d real points. This is because both transversality and the number of real points in an intersection is preserved by small real deformations.

Sections 2 through 4 introduce and develop our basic notions and techniques. Subsequent sections elaborate and apply these ideas. In Section 5, we prove that any enumerative problem on a flag manifold involving five Schubert varieties, three of which are special Schubert varieties, is fully real. Given a smooth map $\pi : Y \to X$, we relate real effective algebraic equivalence on Y to that of X in Section 6 and use this in Sections 7 and 8 to show that many Schubert-type enumerative problems in several classes of flag manifolds are fully real. A proof that enumerative problems involving intersecting hypersurfaces in $\mathbb{P}^n$ are fully real in Section 9 suggests another method for obtaining fully real enumerative problems. This is applied in Section 10 to show that any enumerative problem involving lines incident upon subvarieties of fixed dimension and degree is fully real, and in Section 11 to show the enumerative problem of $(n-2)$-planes in $\mathbb{P}^n$ meeting $2n-2$ rational normal curves is fully real.

2 Intersection Problems

Varieties are reduced, complex, and defined over the real numbers $\mathbb{R}$. Let X and Y denote smooth projective varieties and U, V, and W normal quasi-projective varieties. Equip the real points $X(\mathbb{R})$ of X with the classical topology. Let A^*X be the Chow ring of cycles on X modulo algebraic equivalence.

Two subvarieties meet *generically transversally* if they meet transversally along a dense subset of each component of their intersection. Such an intersection scheme is *generically reduced*, that is, reduced at the generic point of each component. A subvariety $\Xi \subset U \times X$ (or $\Xi \to U$) with generically reduced equidimensional fibres over a normal base U is a family of *multiplicity-free cycles on X over U*. All fibres of Ξ over U are algebraically equivalent, and we say $\Xi \to U$ *represents* that algebraic equivalence class. By a real fibre of Ξ or real member of the family Ξ, we mean a fibre over a point of $U(\mathbb{R})$. Our attention is restricted to multiplicity-free cycles for the simple reason that multiplicities may introduce complex conjugate pairs of solutions, complicating (and perhaps obstructing) our analysis.

These deformation methods may involve comparing cycles from different families. Chow varieties provide a canonical place for such comparisons. For an elaboration of the properties of Chow varieties, see [8] and the references contained therein. Also [2] contains a discussion of Chow varieties in the analytic category, which suffices for our purposes. Positive cycles on X of a fixed dimension and degree are parameterized by a Chow variety of X. We suppress the dependence on dimension and degree and write $\mathit{Chow}\,X$ for any Chow variety of X. The open Chow variety $\mathit{Chow}^\circ X$ is the open subset of $\mathit{Chow}\,X$ parameterizing multiplicity-free cycles on X. There is a tautological family $\Phi \to Chow^\circ X$ of cycles on X with the property that $\zeta \in Chow^\circ X$ represents the the fundamental cycle of the fibre Φ_ζ. Moreover, Φ extends to a family Φ over all of $Chow\,X$, where the cycle represented by $\zeta \in Chow\,X$ has support equal to the support of the fibre Φ_ζ. Furthermore, families $\Xi \to U$ of multiplicity-free cycles on X over U are classified by algebraic morphisms from U to $Chow^\circ X$, since U is normal [8]. Let ϕ be the map classifying such a family $\Xi \to U$. Then for $u \in U$, $\phi(u)$ is the point on $Chow\,X$ representing the fundamental cycle of the fibre Ξ_u.

Two families $\Xi \to U$ and $\Psi \to V$ of multiplicity-free cycles on X are *equivalent* if they have essentially the same cycles. That is, if $\overline{\phi(U)} = \overline{\phi'(V)}$, where ϕ and ϕ' are the maps classifying Ξ and Ψ, respectively. Our results remain valid when one family of cycles is replaced by an equivalent family, perhaps with the additional assumption that $\overline{\phi(U(\mathbb{R}))} = \overline{\phi(V(\mathbb{R}))}$. The varieties $Chow\,X$ and $Chow^\circ X$ as well as the classifying map $\phi : U \to Chow\,X$ are defined over $\mathbb{R}$ [19, §I.9]. We shall let ϕ denote the classifying map of whichever family we are considering.

We consider intersections which vary within families. Suppose that for each $1 \le i \le b$, $\Xi_i \to U_i$ is a family of multiplicity-free cycles on X. Then $\prod_{i=1}^b U_i$ parameterizes all possible intersections of fibres from the families $\Xi_1, \ldots, \Xi_b$. If there is a non-empty locus $U \subset \prod_{i=1}^b U_i$ of points where the intersection is generically transverse, then we say that the families $\Xi_1, \ldots, \Xi_b$ constitute a *(well-posed) intersection problem*. Set $\Xi \to U$ to be the resulting family of intersection cycles, a multiplicity-free family of cycles on X. We shall often neglect the dependence on the original families and refer to $\Xi \to U$ as this intersection problem.

Not all collections of families of cycles give well-posed intersection problems, some transversality is needed to guarantee U is nonempty. When a reductive group acts transitively on X, Kleiman's Transversality Theorem [14] has the following consequence:

Proposition 1 *Suppose a reductive group acts transitively on X, Ξ_1 is a constant family, and for each $2 \le i \le b$, Ξ_i is equivalent to a family of multiplicity-free cycles stable under that action. Then $\Xi_1, \ldots, \Xi_b$ give a well-posed intersection problem.*

Grassmannians and flag manifolds have such an action. For these, we suppose all families of cycles are stable under that action, and thus give well-posed intersection problems.

Suppose a reductive group acts on X with a dense open orbit X'. This occurs if, for instance, X is a toric variety, or more generally, a spherical variety [4,15,17]. Each family may be stable under that action, but the collection need not give a well-posed intersection problem as Kleiman's Theorem only guarantees transversality in X'. However, it is often the case that only points of intersection in X' are desired, and suitable blow up of X or a different equivariant compactification of X' exists on which the corresponding intersection problem is well-posed (see, for example [10, §1.4] or [9, §9 and §10.4]).

3 Effective Algebraic Equivalence

Let $\alpha_1, \ldots, \alpha_a$ be distinct additive generators of A^*X, and for $1 \leq i \leq a$, suppose $\Psi(\alpha_i) \to V(\alpha_i)$ is a family of multiplicity-free cycles on X representing the cycle class α_i. When X is a Grassmannian or flag manifold, $\alpha_1, \ldots, \alpha_a$ will be the Schubert classes and $\Psi(\alpha_i) \to V(\alpha_i)$ will be the corresponding families of Schubert varieties.

We make precise the notion of real effective algebraic equivalence of §1. A family of multiplicity-free cycles $\Xi \subset U \times X$ with classifying map ϕ has *effective algebraic equivalence* with *witness* $Z \in \overline{\phi(U)} \cap Chow^\circ X$ if each (necessarily multiplicity-free) component of Z is a fibre of some family $\Psi(\alpha_i)$. This effective algebraic equivalence is *real* if additionally $Z \in \overline{\phi(U(\mathbb{R}))}$ and each component of Z is a real fibre of some family $\Psi(\alpha_i)$ (a fibre over $V(\alpha_i)(\mathbb{R})$). An intersection problem $\Xi_1, \ldots, \Xi_b$ has *(real) effective algebraic equivalence* if its family of intersection cycles $\Xi \to U$ has (real) effective algebraic equivalence.

Real effective algebraic equivalence has a more intuitive formulation: A cycle Z as above witnesses real effective algebraic equivalence for a family $\Xi \to U$ if and only if Z is a deformation of real fibres of Ξ. Specifically, we have:

Proposition 2 *A multiplicity-free cycle Z whose components are fibres of the families $\Psi(\alpha_i)$ witnesses real effective algebraic equivalence for a family $\Xi \to U$ if and only if there is a family $\Phi \to W$ over a normal base W with a connected subset S of $W(\mathbb{R})$ such that Z is a fibre of $\Phi|_S$ and general fibres of Φ over S are equal (as cycles) to real fibres of Ξ.*

PROOF. For the forward implication, suppose Z witnesses real effective algebraic equivalence for the family $\Xi \to U$. Let W be the normalization of $\overline{\phi(U)} \bigcap Chow^\circ X$, the family Φ the pullback of the tautological family over $Chow^\circ X$, and S the inverse image of $\overline{\phi(U(\mathbb{R}))}$ in W.

For the other implication, let ϕ' be the classifying map of $\Phi \to W$. Then $Z \in \overline{\phi'(S)} \subset \overline{\phi(U(\mathbb{R}))}$, which shows that Z witnesses real effective algebraic equivalence for $\Xi \to U$. $\square$

Let $\beta_1, \ldots, \beta_b$ be classes from $\{\alpha_1, \ldots, \alpha_a\}$. Suppose the families $\Psi(\beta_1), \ldots, \Psi(\beta_b)$ give an intersection problem $\Psi \to V$ having effective algebraic equivalence with witness Z. Fibres of $\Psi \to V$ are generically transverse intersections of fibres of $\Psi(\beta_1), \ldots, \Psi(\beta_b)$, and so have cycle class $\prod_{i=1}^{b} \beta_i$. As $Z \in \overline{\phi(V)}$, this product equals the cycle class of Z, which is $\sum_{i=1}^{a} c_i \cdot \alpha_i$, where c_i counts the components of Z lying in the family $\Psi(\alpha_i)$. Thus in A^*X, we have

$$\prod_{i=1}^{b} \beta_i \;=\; \sum_{i=1}^{a} c_i \,\cdot\, \alpha_i.$$

To compute products in A^*X, classical geometers would try to understand a generically transverse intersection of degenerate cycles in special position, as a generic intersection cycle is typically too difficult to describe. Effective algebraic equivalence extends this method of degeneration by also considering limiting positions of such intersection cycles as the subvarieties degenerate further, even to the point of attaining excess intersection.

A Schubert subvariety $\Omega_w F_\bullet$ of a flag manifold is determined by a complete flag $F_\bullet$ and a coset w of a parabolic subgroup in the symmetric group. Thus Schubert classes σ_w are indexed by these cosets and families Ψ_w of Schubert varieties have base $\mathbb{F}\ell$, the manifold of complete flags.

A *special Schubert subvariety* of a Grassmannian is the locus of planes in $\mathbb{P}^n$ whose intersection with a fixed linear subspace has dimension exceeding the expected dimension. More generally, a special Schubert subvariety of a flag manifold is the pullback of a special Schubert subvariety from a Grassmannian projection. If m is the index of a special Schubert class, then the Pieri-type formula for flag manifolds [16,20] shows that for any w, there exists a subset $I_{m,w}$ of these cosets such that

$$\sigma_m \cdot \sigma_w = \sum_{x \in I_{m,w}} \sigma_x. \tag{2}$$

Theorem 3 *Let w and m be indices of Schubert varieties in a flag manifold, with m the index of a special Schubert variety. Then the intersection problem $\Xi \to U$ given by the families Ψ_w and Ψ_m has real effective algebraic equivalence with witness $\sum_{x \in I_{m,w}} \Omega_x F_\bullet$, where $F_\bullet$ is a real complete flag.*

PROOF. The Borel subgroup B of $\mathrm{GL}_n\mathbb{C}$ stabilizing a real complete flag $F_\bullet$ acts on the Chow variety with fixed points the B-stable cycles, which are sums of Schubert varieties determined by $F_\bullet$. As Hirschowitz [11] observed, $\overline{\phi(U)}$ is B-stable, and must contain a fixed point [3, III.10.4]. In fact, if $F_\bullet'$ is a real flag in linear general position with $F_\bullet$, then the $B(\mathbb{R})$-orbit of $\Omega_m F_\bullet \bigcap \Omega_w F_\bullet'$ is a subset of $\phi(U(\mathbb{R}))$. Moreover, its closure has a $B(\mathbb{R})$-fixed point, as the proof in [3] may be adapted to show that complete $B(\mathbb{R})$-stable real analytic sets have fixed points. Since the coefficients of the sum (2) are all 1, $\sum_{x \in I_{m,w}} \Omega_x F_\bullet$ is the only $B(\mathbb{R})$-stable cycle in its algebraic equivalence class, and therefore

$$\sum_{x \in I_{m,w}} \Omega_x F_\bullet \in \overline{\phi(U(\mathbb{R}))}. \qquad \square$$

4 Fully Real Enumerative Problems

An *enumerative problem* of *degree* d is an intersection problem $\Xi \to U$ with zero-dimensional fibres of cardinality d. An enumerative problem is *fully real* if there exists $u \in U(\mathbb{R})$ with all points in the fibre Ξ_u real. In this case, $u = (u_1, \ldots, u_b)$ with each $u_i \in U_i(\mathbb{R})$ and the cycles $(\Xi_1)_{u_1}, \ldots, (\Xi_b)_{u_b}$ meet transversally with all points of intersection real.

Theorem 4 *An enumerative problem $\Xi \to U$ is fully real if and only if it has real effective algebraic equivalence. That is, if and only if there exists a point $\zeta \in \overline{\phi(U(\mathbb{R}))}$ representing distinct real points.*

PROOF. The forward implication is a consequence of the definition. For the reverse, let d be the degree of $\Xi \to U$ and ϕ its classifying map. Then $\phi : U \to S^d X$, the Chow variety of effective degree d zero-cycles on X. The real points $S^d X(\mathbb{R})$ of the Chow variety represent degree d zero-cycles stable under complex conjugation. Its dense set of multiplicity-free cycles has an open subset $\mathcal{M}$ parameterizing cycles of distinct real points, and $\zeta \in \mathcal{M}$. Thus $\phi(U(\mathbb{R})) \cap \mathcal{M} \neq \emptyset$, which implies $\Xi \to U$ is fully real. $\square$

5 Pieri-Type Enumerative Problems

Theorem 5 *Any enumerative problem in any flag manifold involving five Schubert varieties, three of which are special, is fully real.*

PROOF. Let w_1, w_2 be indices of Schubert varieties and m_1, m_2, m_3 indices of special Schubert varieties in a flag manifold. Suppose the families Ψ_{w_1}, Ψ_{w_2}, Ψ_{m_1}, Ψ_{m_2}, and Ψ_{m_3} give an enumerative problem $\Xi \to U$.

By Theorem 3, for each $i = 1, 2$, the intersection problem $\Gamma_i \to V_i$ given by the families Ψ_{w_i} and Ψ_{m_i} has real effective algebraic equivalence with witness $\sum_{x_i \in I_{m_i, w_i}} \Omega_{x_i} F_\bullet$, for any real flag $F_\bullet$. Let $F_\bullet$ and $F_\bullet'$ be real flags in general position and set

$$Z_1 := \sum_{x_1 \in I_{m_1, w_1}} \Omega_{x_1} F_\bullet \qquad \text{and} \qquad Z_2 := \sum_{x_2 \in I_{m_2, w_2}} \Omega_{x_2} F_\bullet'.$$

Since $F_\bullet$ and $F_\bullet'$ are in general position, $Z_1 \bigcap Z_2$ is a generically transverse intersection. By Kleiman's Transversality Theorem [14] there exists a real flag $E_\bullet$ such that $Z_1 \bigcap Z_2 \bigcap \Omega_{m_3} E_\bullet$ is a transverse intersection. Components of $Z_1 \bigcap Z_2$ are intersections $\Omega_{x_1} F_\bullet \bigcap \Omega_{x_2} F_\bullet'$ of two Schubert varieties. By the Pieri-type formula for flag manifolds [16,20], each triple intersection

$$\Omega_{x_1} F_\bullet \bigcap \Omega_{x_2} F_\bullet' \bigcap \Omega_{m_3} E_\bullet$$

either is empty, or is a single (necessarily) real point. Thus $Z_1 \bigcap Z_2 \bigcap \Omega_{m_3} E_\bullet$ is a transverse intersection all of whose points are real. For $i = 1, 2$, Z_i is a deformation of real cycles of the family Γ_i. This implies there exist real fibres $(\Gamma_1)_{v_1}$ and $(\Gamma_2)_{v_2}$ of the families Γ_1 and Γ_2 such that $(\Gamma_1)_{v_1} \bigcap (\Gamma_2)_{v_2} \bigcap \Omega_{m_3} E_\bullet$ is a transverse intersection all of whose points are real, as transversality and the number of real points in an intersection are preserved by small real deformations. The fibre $(\Gamma_1)_{v_1}$ is a generically transverse intersection of real cycles from Ψ_{w_1} and Ψ_{m_1}. Likewise, $(\Gamma_2)_{v_2}$ is a generically transverse intersection of real cycles from Ψ_{w_2} and Ψ_{m_2}. In other words, $\Xi \to U$ is fully real. $\square$

6 Fibrations

Suppose $\pi : Y \to X$ is a smooth morphism. If $\Xi \to U$ is a family of multiplicity-free cycles on X representing the cycle class α, its pullback $\pi^*\Xi := (1_U \times \pi)^{-1}\Xi \to U$ is a family of multiplicity-free cycles on Y representing the cycle class $\pi^*\alpha$.

Suppose $\alpha_1 \ldots, \alpha_a$ generate A^*X additively and $\Psi(\alpha_1), \ldots, \Psi(\alpha_a)$ are families of cycles representing these generators. The classes $\pi^*\alpha_1, \ldots, \pi^*\alpha_a$ span the image of A^*X in A^*Y and are represented by the families $\pi^*\Psi(\alpha_1), \ldots, \pi^*\Psi(\alpha_a)$. Effective algebraic equivalence is preserved by pullbacks:

Theorem 6 *If $\Xi \to U$ is a family of multiplicity-free cycles on X having effective algebraic equivalence with witness Z, then $\pi^*\Xi \to U$ is a family of multiplicity-free cycles on Y having effective algebraic equivalence with witness $\pi^{-1}Z$. Likewise, if $\Xi \to U$ has real effective algebraic equivalence, then so does $\pi^*\Xi \to U$.*

PROOF. If $\Xi \to U$ has an effective algebraic equivalence with witness Z, then there is a family $\Phi \to W$ of cycles containing Z whose general member is a fibre of Ξ. Thus the family $\pi^*\Phi \to W$ contains $\pi^{-1}Z$ and its general member is a fibre of $\pi^*\Xi$.

If $\Xi \to U$ has real effective algebraic equivalence with witness Z, then $W(\mathbb{R})$ has a connected subset S and Z is a fibre of Φ over a point of S. Again, considering $\pi^*\Phi$ shows $\pi^*\Xi$ has real effective algebraic equivalence with witness $\pi^{-1}Z$. $\square$

7 Schubert-Type Enumerative Problems in $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ Are Fully Real

The variety $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ of partial flags $q \in l \subset \mathbb{P}^n$ where q is a point and l a line has projections

$$p : \mathbb{F}\ell_{0,1}\mathbb{P}^n \longrightarrow \mathbb{P}^n \qquad \text{and} \qquad \pi : \mathbb{F}\ell_{0,1}\mathbb{P}^n \longrightarrow \mathbb{G}_1\mathbb{P}^n,$$

where $\mathbb{G}_1\mathbb{P}^n$ is the Grassmannian of lines in $\mathbb{P}^n$.

A Schubert subvariety $\Omega(F, P)$ of $\mathbb{G}_1\mathbb{P}^n$ is determined by a partial flag $F \subset P$ of $\mathbb{P}^n$:

$$\Omega(F, P) \ := \ \{l \in \mathbb{G}_1\mathbb{P}^n \mid l \cap F \neq \emptyset \ \text{ and } \ l \subset P\}.$$

If F is a hyperplane of P, then $\Omega(F,P) = \mathbb{G}_1 P$, the Grassmannian of lines in P.

In addition to $\pi^{-1}\Omega(F,P)$, there is one other Schubert subvariety of $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ which projects onto $\Omega(F,P)$ in $\mathbb{G}_1\mathbb{P}^n$:

$$\widehat{\Omega}(F,P) \ := \ \{(q,l) \in \mathbb{F}\ell_{0,1}\mathbb{P}^n \mid q \in F \ \text{ and } \ l \subset P\}.$$

Any Schubert subvariety of $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ is one of $\Omega(F,P)$ or $\widehat{\Omega}(F,P)$, for suitable $F \subset P$. The varieties $\widehat{\Omega}(F,P)$ have another description, which is straightforward to verify:

Lemma 7 *Let N, P be subspaces of $\mathbb{P}^n$. Then*

$$p^{-1}N \bigcap \pi^{-1}\mathbb{G}_1 P \ = \ \widehat{\Omega}(N \cap P,\, P),$$

and, if N and P meet properly, this intersection is generically transverse.

Corollary 8 *Any Schubert-type enumerative problem on $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ is equivalent to one involving only pullbacks of Schubert subvarieties from $\mathbb{P}^n$ and $\mathbb{G}_1\mathbb{P}^n$.*

The next lemma, an exercise in linear algebra, describes Poincaré duality for Schubert subvarieties of $\mathbb{F}\ell_{0,1}\mathbb{P}^n$.

Lemma 9 *Suppose a linear subspace N meets a partial flag $F \subset P$ properly in $\mathbb{P}^n$. If $\pi^{-1}\Omega(F,P)$ and $p^{-1}N$ have complimentary dimension in $\mathbb{F}\ell_{0,1}\mathbb{P}^n$, then their intersection is empty unless both $N \cap P$ and F are points. In that case, $\pi^{-1}\Omega(F,P)$ and $p^{-1}N$ meet transversally in the single partial flag $N \cap P \ \in \ \langle N \cap P, F\rangle$, which is a point of $\mathbb{F}\ell_{0,1}\mathbb{P}^n$.*

Theorem 10 *Any Schubert-type enumerative problem in $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ is fully real.*

PROOF. By Corollary 8, it suffices to consider enumerative problems involving only pullbacks of Schubert subvarieties from $\mathbb{P}^n$ and $\mathbb{G}_1\mathbb{P}^n$. Since an intersection of linear subspaces in $\mathbb{P}^n$ is another linear subspace, we may further suppose the enumerative problem $\Xi \to U$ is given by families $p^*\Xi_1, \pi^*\Xi_2, \ldots, \pi^*\Xi_b$, where Ξ_1 is a family of subspaces of a fixed dimension in $\mathbb{P}^n$ and $\Xi_2, \ldots, \Xi_b$ are families of Schubert subvarieties of $\mathbb{G}_1\mathbb{P}^n$.

The intersection problem $\Psi \to V$ on $\mathbb{G}_1\mathbb{P}^n$ given by $\Xi_2, \ldots, \Xi_b$ has real effective algebraic equivalence [21, Theorem C]. Let Z be a witness. By Theorem 6, $\pi^*\Psi \to V$ has real effective algebraic equivalence with witness π^*Z.

Let $\mathcal{L}$ be the lattice of subspaces of $\mathbb{P}^n$ generated by the (necessarily real) subspaces defining components of Z, and let N be a real subspace from the family Ξ_1 meeting all subspaces of $\mathcal{L}$ properly. By Lemma 9, $p^{-1}N \bigcap \pi^{-1}Z$ is transverse with all points of intersection real. Since $\pi^{-1}Z$ is a deformation of real cycles of the family $\pi^*\Psi$ and both transversality and the number of real points in an intersection are preserved by small real deformations, there is a real fibre $(\pi^*\Psi)_v$ of $\pi^*\Psi$ such that $p^{-1}N\bigcap(\pi^*\Psi)_v$ is transverse with all points of intersection real. Since $(\pi^*\Psi)_v$ is a generically transverse intersection of real Schubert varieties from the families $\pi^*\Xi_2,\ldots,\pi^*\Xi_b$, the enumerative problem $\Xi\to U$ is fully real. $\square$

Theorem 11 *Any Schubert-type intersection problem on $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ has real effective algebraic equivalence.*

We give an outline, as a complete analysis involves no new ideas beyond those of [21].

By Corollary 8, it suffices to consider intersection problems $\Xi\to U$ given by families $p^*\Xi_1$, $\pi^*\Xi_2,\ldots,\pi^*\Xi_b$, where Ξ_1 is a family of subspaces of a fixed dimension in $\mathbb{P}^n$ and $\Xi_2,\ldots,\Xi_b$ are families of Schubert subvarieties of $\mathbb{G}_1\mathbb{P}^n$.

The intersection problem given by $\Xi_2,\ldots,\Xi_b$ has real effective algebraic equivalence [21, Theorem C]. Let Z be a witness. Let $\Psi\to V$ be the intersection problem given by $p^*\Xi_1$ and the constant family $\pi^{-1}Z$. Since $\pi^{-1}Z$ is a deformation of intersections of real cycles from the families $\pi^*\Xi_2,\ldots,\pi^*\Xi_b$, we have

$$\phi(V)\subset\overline{\phi(U)}\qquad\text{and}\qquad\phi(V(\mathbb{R}))\subset\overline{\phi(U(\mathbb{R}))}.$$

Thus, it suffices to show $\Psi\to V$ has real effective algebraic equivalence.

A proof that $\Psi\to V$ has real effective algebraic equivalence mimics the proof of the corresponding result [21, Theorem A] for $\mathbb{G}_1\mathbb{P}^n$, with the following lemma playing the role of Lemma 2.4 of [21].

Lemma 12 *Let $H\subset\mathbb{P}^n$ be a hyperplane, $P\not\subset H$ a linear subspace, and $F\subset P\cap H$ a proper linear subspace. Let $N\not\subset H$ be a linear subspace meeting F — and hence P — properly, and set $L=N\cap H$. Then $\pi^{-1}\Omega(F,P)$ and $p^{-1}L$ meet generically transversally,*

$$\pi^{-1}\Omega(F,P)\bigcap p^{-1}L\;=\;\widehat{\Omega}(N\cap F,\,P)\;+\;\pi^{-1}\Omega(F,P\cap H)\bigcap p^{-1}N,$$

and the second term is itself an irreducible generically transverse intersection.

The proof of this statement closely parallels the proof of Lemma 2.4 of [21].

8 Some Schubert-Type Enumerative Problems in $\mathbb{F}\ell_{1,n-2}\mathbb{P}^n$

The manifold $\mathbb{F}\ell_{1,n-2}\mathbb{P}^n$ of partial flags $l\subset\Lambda\subset\mathbb{P}^n$, where l is a line and Λ an $(n-2)$-plane, has natural projections

$$\pi:\mathbb{F}\ell_{1,n-2}\mathbb{P}^n\to\mathbb{G}_1\mathbb{P}^n\qquad\text{and}\qquad p:\mathbb{F}\ell_{1,n-2}\mathbb{P}^n\to\mathbb{G}_{n-2}\mathbb{P}^n,$$

where $\mathbb{G}_{n-2}\mathbb{P}^n$ is the Grassmannian of $(n-2)$-planes in $\mathbb{P}^n$.

Theorem 13 *Any enumerative problem in $\mathbb{F}\ell_{1,n-2}\mathbb{P}^n$ given by pullbacks of Schubert subvarieties from $\mathbb{G}_1\mathbb{P}^n$ and $\mathbb{G}_{n-2}\mathbb{P}^n$ is fully real.*

PROOF. Suppose $\pi^*\Xi_1, \ldots, \pi^*\Xi_b, p^*\Gamma_1, \ldots, p^*\Gamma_c$ give an enumerative problem on the manifold $\mathbb{F}\ell_{1,n-2}\mathbb{P}^n$ where, for $1 \leq i \leq b$, Ξ_i is a family of Schubert subvarieties of $\mathbb{G}_1\mathbb{P}^n$ and for $1 \leq j \leq c$, Γ_j is a family of Schubert subvarieties of $\mathbb{G}_{n-2}\mathbb{P}^n$.

By Theorem 6 and Theorem C of [21], $\pi^*\Xi_1, \ldots, \pi^*\Xi_b$ give an intersection problem $\Psi_1 \to V_1$ which has real algebraic equivalence. Let Z_1 be a witness. Identifying $\mathbb{P}^n$ with its dual projective space gives an isomorphism $\mathbb{G}_{n-2}\mathbb{P}^n \xrightarrow{\sim} \mathbb{G}_1\mathbb{P}^n$, mapping Schubert subvarieties to Schubert subvarieties. It follows that $p^*\Gamma_1, \ldots, p^*\Gamma_c$ give an intersection problem $\Psi_2 \to V_2$ which has real algebraic equivalence. Let Z_2 be a witness. It suffices to show the enumerative problem given by Ψ_1 and Ψ_2 is fully real.

Since Z_1 and Z_2 may be replaced by any translate by elements of $\mathrm{PGL_{n+1}}\mathbb{R}$ (which is Zariski-dense in $\mathrm{PGL_{n+1}}\mathbb{C}$), we may assume Z_1 and Z_2 intersect transversally, by Kleiman's Transversality Theorem [14]. Components of Z_1 and Z_2 are Schubert varieties defined by real flags. Moreover, each component of Z_1 has complementary dimension to each component of Z_2. In a flag manifold, Schubert varieties of complimentary dimension which meet transversally and are defined by real flags either have empty intersection, or meet in a single real point. (Lemma 9 was a particular case.) Thus $Z_1 \bigcap Z_2$ consists entirely of real points.

Each cycle Z_i is a deformation of real cycles from the family Ψ_i. Since both transversality and the number of real points in an intersection are preserved by small real deformations, there exists real fibres $(\Psi_1)_{v_1}$ and $(\Psi_2)_{v_2}$ of the families Ψ_1 and Ψ_2 such that $(\Psi_1)_{v_1} \bigcap (\Psi_2)_{v_2}$ is transverse and consists entirely of real points. Thus the enumerative problem given by Ψ_1 and Ψ_2, and hence the original problem, is fully real. $\square$

One may similarly use Theorem 11 to prove an analogous result for $\mathbb{F}\ell_{0,1,n-2,n-1}\mathbb{P}^n$, the manifold of partial flags $p \in \ell \subset \Lambda \subset H \subset \mathbb{P}^n$, where p is a point, ℓ a line, Λ an $(n-2)$-plane, and H a hyperplane in $\mathbb{P}^n$. When $n = 4$, this is the manifold of complete flags in $\mathbb{P}^4$:

Theorem 14 *Any enumerative problem in $\mathbb{F}\ell_{0,1,n-2,n-1}\mathbb{P}^n$ given by pullbacks of Schubert varieties from $\mathbb{F}\ell_{0,1}\mathbb{P}^n$ and $\mathbb{F}\ell_{n-2,n-1}\mathbb{P}^n$ is fully real.*

9 Powers of Enumerative Problems

A method to construct a new fully real enumerative problem out of a given one is illustrated by the following proposition about intersecting hypersurfaces in $\mathbb{P}^n$. We will exploit this method in subsequent sections.

Proposition 15 *Let $d_1, \ldots, d_n$ be positive integers. Then there exist smooth real hypersurfaces $D_1, \ldots, D_n$ in $\mathbb{P}^n$ of respective degrees $d_1, \ldots, d_n$ which meet transversally in $\prod_{i=1}^n d_i$ real points.*

PROOF. This is a consequence of the following observations: First, any enumerative problem given by intersecting hyperplanes in $\mathbb{P}^n$ is fully real, from which it follows that any enumerative problem given by intersecting cycles composed of unions of distinct hyperplanes is fully real. Second, real hypersurfaces may be deformed into unions of distinct real hyperplanes. Lastly, both transversality and the number of real points of intersection are preserved by small real deformations. $\square$

We generalize (and formalize) the first observation: Suppose $\Xi \to U$ is a family of multiplicity-free cycles on X and d is a positive integer. Let $U^{(d)}$ be the locus of d-tuples $(u_1, \ldots, u_d) \in U^d$ such that no two of the fibres $\Xi_{u_1}, \ldots, \Xi_{u_d}$ share a component. Define $\Xi^{\oplus d} \to U^{(d)}$ to be the family of multiplicity-free cycles whose fibre over $(u_1, \ldots, u_d) \in U^{(d)}$ is $\sum_{j=1}^d \Xi_{u_j}$.

Suppose $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ are families of multiplicity-free cycles on X giving an intersection problem $\Xi \to U$ and $d_1, \ldots, d_b$ is a sequence of positive integers. Then the families $\Xi_1^{\oplus d_1} \to U_1^{(d_1)}, \ldots, \Xi_b^{\oplus d_b} \to U_b^{(d_b)}$ give a well-posed intersection problem whenever each $U_i^{(d_i)}$ is nonempty.

When a reductive group G acts transitively on X and each family Ξ_i of cycles is G-stable, the families $\Xi_1^{\oplus d_i}, \ldots, \Xi_b^{\oplus d_b}$ give an intersection problem, without any additional hypotheses. Moreover, if $\Xi \to U$ is a fully real enumerative problem, then so is the enumerative problem given by $\Xi_1^{\oplus d_i}, \ldots, \Xi_b^{\oplus d_b}$. We produce a witness with a particular form, which will be useful in §11.

Lemma 16 *Suppose $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ give a fully real enumerative problem of degree d. Let $d_1, \ldots, d_b$ be a sequence of positive integers and suppose that for $1 \leq i \leq b$, V_i is a G-stable subset of $U_i^{(d_i)}$ such that the diagonal $\Delta^{d_i} U_i(\mathbb{R}) \subset \overline{V_i(\mathbb{R})}$, as subsets of $U_i(\mathbb{R})^{d_i}$. Then for each $1 \leq i \leq b$, there exists a point $v_i \in V_i(\mathbb{R})$ such that the cycles $(\Xi_1^{\oplus d_1})_{v_1}, \ldots, (\Xi_b^{\oplus d_b})_{v_b}$ intersect transversally in $d \cdot \prod_{i=1}^b d_i$ real points.*

PROOF. The restriction Ψ_i of $\Xi_i^{\oplus d_i}$ to V_i is a G-stable family. Thus $\Psi_1, \ldots, \Psi_b$ give a well-posed enumerative problem $\Psi \to V$. We show this is fully real and compute its degree.

Let $\Xi \to U$ be the enumerative problem given by the families $\Xi_1, \ldots, \Xi_b$. Since $\Xi \to U$ is fully real, there is an open subset M of $U(\mathbb{R})$ consisting of points $u \in U(\mathbb{R})$ such that Ξ_u is d distinct real points. Recall that $U(\mathbb{R})$ is an open subset of the product $\prod_{i=1}^{b} U_i(\mathbb{R})$. Thus, for each $1 \leq i \leq b$, there exists an open subset M_i of $U_i(\mathbb{R})$ such that $\prod_{i=1}^{b} M_i \subset M$. Note that $\Delta^{d_i} M_i \subset \Delta^{d_i} U_i(\mathbb{R}) \subset \overline{V_i(\mathbb{R})}$, from which it follows that $V_i(\mathbb{R}) \bigcap M_i^{d_i}$ is nonempty and open in $V_i(\mathbb{R})$. Thus $M' := V(\mathbb{R}) \bigcap \prod_{i=1}^{b} M_i^{d_i}$ is nonempty, as $V(\mathbb{R})$ is dense in $\prod_{i=1}^{b} V_i(\mathbb{R})$.

Let $w = (w_{11}, \ldots, w_{1d_1}, \ldots, w_{b1} \ldots, w_{bd_b}) \in M'$. From the construction of M', each $w_{ij} \in M_i$ and each d_i-tuple $(w_{i1}, \ldots, w_{id_i}) \in V_i(\mathbb{R})$. Also, given any sequence $j_1, \ldots, j_b$ satisfying $1 \leq j_i \leq d_i$ for $1 \leq i \leq b$, the b-tuple $(w_{1j_1}, \ldots, w_{bj_b}) \in U(\mathbb{R})$. Furthermore,

$$\Psi_w = \bigcap_{i=1}^{b} (\Psi_i)_{(w_{i1}, \ldots, w_{id_i})}$$

is a transverse intersection, as $M' \subset V$. Since $(\Psi_i)_{(w_{i1}, \ldots, w_{id_i})} = \sum_{j=1}^{d_i} (\Xi_i)_{w_{ij}}$, we have

$$\Psi_w \;=\; \bigcap_{i=1}^{b} \sum_{j=1}^{d_i} (\Xi_i)_{w_{ij}} \;=\; \sum_{\substack{j_1, \ldots, j_b \\ 1 \leq j_i \leq d_i}} \bigcap_{i=1}^{b} (\Xi_i)_{w_{ij_i}} \;=\; \sum_{\substack{j_1, \ldots, j_b \\ 1 \leq j_i \leq d_i}} \Xi_{(w_{1j_1}, \ldots, w_{bj_b})}.$$

Since the intersection defining Ψ_w is transverse, the last equality shows it consists of $d \cdot \prod_{i=1}^{b} d_i$ real points. $\square$

10 Lines in $\mathbb{P}^n$ Meeting Real Subvarieties

As an application of Lemma 16, we show that any enumerative problem involving lines in $\mathbb{P}^n$ meeting real subvarieties of fixed dimension and degree is fully real. Specifically, we prove:

Theorem 17 *Let $a_1, \ldots, a_b$ be positive integers with $a_1 + \cdots + a_b = 2n - 2$. Then for any positive integers $d_1, \ldots, d_b$, the enumerative problem of lines meeting smooth subvarieties $X_1, \ldots, X_b$, where X_i has dimension $n - a_i - 1$ and degree d_i, is fully real.*

PROOF. When the degrees d_i are all 1, so that X_i is a linear subspace of dimension $n - a_i - 1$, this is just Theorem C of [21].

Consider now the general case of arbitrary positive degrees. By Lemma 16, the enumerative problem given by lines meeting cycles $Y_1, \ldots, Y_b$ is fully real, where each Y_i is a union of d_i distinct $(n - a_i - 1)$-planes. Let $Y_1, \ldots, Y_b$ witness this enumerative problem being fully real. Since each Y_i is a real deformation of smooth subvarieties of degree d_i and dimension $n-a_i-1$, choosing such subvarieties X_i sufficiently close to each Y_i shows the original enumerative problem is fully real. $\square$

11 $(n-2)$-Planes Meeting Rational Normal Curves in $\mathbb{P}^n$

Let $\mathbb{G}_{n-2}\mathbb{P}^n$ be the Grassmannian of $(n-2)$-planes in $\mathbb{P}^n$, a variety of dimension $2n-2$. Those $(n-2)$-planes which meet a curve form a hypersurface in $\mathbb{G}_{n-2}\mathbb{P}^n$. We synthesize ideas of previous sections to prove the following theorem.

Theorem 18 *The enumerative problem of $(n-2)$-planes meeting $2n-2$ general rational normal curves in $\mathbb{P}^n$ is fully real and has degree $\binom{2n-2}{n-1}n^{2n-3}$.*

PROOF. Identifying $\mathbb{P}^n$ with its dual projective space gives an isomorphism $\mathbb{G}_{n-2}\mathbb{P}^n \xrightarrow{\sim} \mathbb{G}_1\mathbb{P}^n$, mapping Schubert subvarieties to Schubert subvarieties. Let $\Xi \subset \mathbb{G}_1\mathbb{P}^n \times \mathbb{G}_{n-2}\mathbb{P}^n$ be the correspondence of lines incident upon $(n-2)$-planes. The fibre of Ξ over a point $l \in \mathbb{G}_1\mathbb{P}^n$ is the hypersurface Schubert variety Ω_l of $(n-2)$-planes meeting l.

Any enumerative problem involving Schubert subvarieties of $\mathbb{G}_{n-2}\mathbb{P}^n$ is fully real [21, Theorem C]. In particular, the enumerative problem given by $2n-2$ copies of the family $\Xi \to \mathbb{G}_1\mathbb{P}^n$ is fully real. We compute its degree, d. For the rest of this proof, let U denote $\mathbb{G}_1\mathbb{P}^n$.

The image of Ω_l under the isomorphism $\mathbb{G}_{n-2}\mathbb{P}^n \xrightarrow{\sim} \mathbb{G}_1\mathbb{P}^n$ is the Schubert subvariety consisting of lines meeting a fixed $(n-2)$-plane. Thus d is the number of lines meeting $2n-2$ general $(n-2)$-planes in $\mathbb{P}^n$. By Corollary 3.3 of [21], this is the number of (standard) Young tableaux of shape $(n-1, n-1)$, which is $\frac{1}{n}\binom{2n-2}{n-1}$, by the hook length formula of Frame, Robinson, and Thrall [7]. (This is also the degree of $\mathbb{G}_1\mathbb{P}^n$ in its Plücker embedding.)

Let $e_0, \ldots, e_n$ be real points spanning $\mathbb{P}^n$. For $1 \leq i \leq n$, set $l_i := \langle e_{i-1}, e_i\rangle$. Then $\Omega_{l_1} + \cdots + \Omega_{l_n}$ is the fibre of $\Xi^{\oplus n}$ over $(l_1, \ldots, l_n) \in U^{(n)}(\mathbb{R})$. Let $V \subset U^{(n)}$ be the $\mathrm{PGL_{n+1}}\mathbb{C}$-orbit containing the point $(l_1, \ldots, l_n) \in U^{(n)}$. For $t \in [0,1]$ and $1 \leq i \leq n$, define

$$l_i(t) \;:=\; \langle te_{i-1} + (1-t)e_{\overline{i-1}},\, te_i + (1-t)e_{\overline{i}}\rangle,$$

where, for j an integer, $\overline{\jmath} \in \{0,1\}$ is congruent to j modulo 2. Let $\gamma(t) := (l_1(t), \ldots, l_n(t))$. If $t \in (0,1]$, then $\gamma(t) \in V(\mathbb{R})$. Since $\gamma(0) = (l_1, \ldots, l_1) \in \Delta^n U(\mathbb{R})$ and $\Delta^n U(\mathbb{R}) = \mathrm{PGL_{n+1}}\mathbb{R} \cdot \gamma(0)$, it follows that $\Delta^n U(\mathbb{R}) \subset \overline{V(\mathbb{R})}$. Then, by Lemma 16, there exist points $v_1, \ldots, v_{2n-2} \in V(\mathbb{R})$ such that the cycles $\Xi_{v_1}^{\oplus n}, \ldots, \Xi_{v_{2n-2}}^{\oplus n}$ meet transversally in $\binom{2n-2}{n-1} n^{2n-3}$ real points.

Let $p(m) := n \cdot m + 1$, the Hilbert polynomial of a rational normal curve in $\mathbb{P}^n$. Let $\mathcal{H}$ be the open subset of the Hilbert scheme parameterizing reduced schemes with Hilbert polynomial p. Let $\Psi \subset \mathcal{H} \times \mathbb{G}_{n-2}\mathbb{P}^n$ be the family of multiplicity-free cycles on $\mathbb{G}_{n-2}\mathbb{P}^n$ whose fibre over a curve $C \in \mathcal{H}$ is the hypersurface of $(n-2)$-planes meeting C.

Note that p is also the Hilbert polynomial of $l_1 \bigcup \cdots \bigcup l_n$. Let $\lambda \in \mathcal{H}$ be the point representing $l_1 \bigcup \cdots \bigcup l_n$. If V' is the $\mathrm{PGL_{n+1}}\mathbb{C}$-orbit of λ in $\mathcal{H}$, then $\Psi|_{V'} \to V'$ is isomorphic to the family $\Xi^{\oplus n} \to V$, under the obvious isomorphism between V and V'. It follows that the enumerative problem given by $2n-2$ copies of $\Psi \to \mathcal{H}$ is fully real and has degree $\binom{2n-2}{n-1} n^{2n-3}$. Let W be the subset of $\mathcal{H}$ representing rational normal curves. We claim $V'(\mathbb{R}) \subset \overline{W(\mathbb{R})}$, which will complete the proof.

Let $[x_0, \ldots, x_n]$ be homogeneous coordinates for $\mathbb{P}^n$ dual to the basis $e_0, \ldots, e_n$. For $t \in \mathbb{C}$, define the ideal $\mathcal{I}_t$ by

$$\mathcal{I}_t \ := \ (x_i x_j - t x_{i+1} x_{j-1} \,|\, 0 \leq i < j \leq n \text{ and } j - i \geq 2).$$

For $t \neq 0$, $\mathcal{I}_t$ is the ideal of a rational normal curve and $\mathcal{I}_0$ is the ideal of $l_1 \bigcup \cdots \bigcup l_n$.

This family of ideals is flat. Let $\varphi : \mathbb{C} \to \mathcal{H}$ be the map representing this family. Then $\varphi(\mathbb{R} - \{0\}) \subset W(\mathbb{R})$. Noting $\varphi(0) = \lambda$ shows $\lambda \in \overline{W(\mathbb{R})}$. Since $W(\mathbb{R})$ is $\mathrm{PGL_{n+1}}\mathbb{R}$-stable, we conclude that $V'(\mathbb{R}) \subset \overline{W(\mathbb{R})}$. $\square$

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