Extended abstract, submitted to MEGA 05

EXPERIMENTATION AND CONJECTURES IN THE REAL SCHUBERT CALCULUS FOR FLAG MANIFOLDS

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Abstract. The Shapiro conjecture in the real Schubert calculus fails to hold for flag manifolds, but in a very interesting way. We give a refinement of that conjecture for the flag manifold and present massive experimentation that supports this conjecture. We also establish relationships between different cases of the conjecture.

Introduction

The Shapiro conjecture for Grassmannians [19, 14] has driven progress in enumerative real algebraic geometry [21], which is the study of real solutions to geometric problems. It conjectures that a (zero-dimensional) intersection of Schubert subvarieties of a Grassmannian consists entirely of real points—if the Schubert subvarieties are given by flags osculating a real rational normal curve. This particular geometric problem can be interpreted in terms of real linear series on $\mathbb{P}^1$ with prescribed (real) ramification [1, 2], real rational curves in $\mathbb{P}^n$ with real flexes [9], linear systems theory [13], and the Bethe Ansatz and Fuchsian equations [12]. The Shapiro conjecture has implications for all these areas. Massive computational evidence [19, 23] as well as its proof by Eremenko and Gabrielov for Grassmannians of codimension 2 subspaces [4] give compelling evidence for its validity. A local version, that it holds when the Schubert varieties are special (a technical term) and when the points of osculation are sufficiently clustered [18], showed that the special Schubert calculus is fully real (such geometric problems can have all their solutions real). That result was generalized by Vakil [22] (using other methods) to show that the general Schubert calculus on the Grassmannian is fully real.

The original Shapiro conjecture was that such an intersection of Schubert varieties in a *flag manifold* would consist entirely of real points. Unfortunately, this fails for the first non-trivial enumerative problem on a non-Grassmannian flag manifold (see Section 1.3), but in a very interesting way. This failure was first noted in [19, §5] and a more symmetric counterexample was found in [20], where computer experimentation suggested that the conjecture would hold if the points where the flags osculated the rational normal curve satisfied a certain non-crossing condition. Further experimentation led to a precise formulation of this refined non-crossing conjecture in [21]. That conjecture was only valid for two- and three- step flag manifolds, and the further experimentation reported here

Work done at MSRI and supported by NSF grant DMS-9810361.
Computations done on computers purchased with NSF SCREMS grant DMS-0079536.
Work of Sottile was supported by the Clay Mathematical Institute.
This work was supported in part by NSF CAREER grant DMS-0134860.

leads to a version (Conjecture 2.1) for all flag manifolds in which the points of osculation satisfy a monotonicity property.

We systematically investigate the Shapiro conjecture for flag manifolds to gain a deeper understanding both of its failure and of our refinement. Our investigation includes 12 gigaHertz-years of computer experimentation and theorems relating our monotone conjecture for different enumerative problems. Recently, our conjecture was proven by Eremenko, Gabrielov, Shapiro, and Vainshtein [5] for manifolds of flags consisting of a codimension 2 plane lying on a hyperplane.

Here is one open instance of this conjecture, expressed as a system of polynomials in local coordinates for the variety of flags $E_2 \subset E_3$ in 5-space, where $\dim E_i = i$. Let $t, x_1, \ldots, x_8$ be indeterminates, and consider the polynomials

$$f(t;x) \;:=\; \det \begin{bmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ t^4 & t^3 & t^2 & t & 1 \\ 4t^3 & 3t^2 & 2t & 1 & 0 \\ 6t^2 & 3t & 1 & 0 & 0 \end{bmatrix}, \qquad g(t;x) \;:=\; \det \begin{bmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 0 & 0 & 1 & x_7 & x_8 \\ t^4 & t^3 & t^2 & t & 1 \\ 4t^3 & 3t^2 & 2t & 1 & 0 \end{bmatrix}.$$

Conjecture A. *Let $t_1 < t_2 < \cdots < t_8$ be real numbers. Then the polynomial system*

$$\begin{aligned} f(t_1;x) = f(t_2;x) = f(t_3;x) = f(t_4;x) &= 0, \quad \textit{and} \\ g(t_5;x) = g(t_6;x) = g(t_7;x) = g(t_8;x) &= 0 \end{aligned}$$

has 12 solutions, and all of them are real.

Monotonicity is that the polynomials f are evaluated at parameter values that are less than the parameter values at which the polynomials g are evaluated. If the order of t_4 and t_5 were switched, then the evaluation would not be monotone. We computed 100,000 instances of this polynomial system at different ordered parameter values, and each had 12 real solutions. In contrast, we found non-monotone evaluations for which not all solutions were real, and the minimum number of real solutions that we observed depended on the combinatorics of the evaluation. This is summarized in Table 1.

Section 1 contains background on flag manifolds, states the Shapiro Conjecture, and gives a geometrically vivid example of its failure. In Section 2, we state our conjectures and discuss relations among them. The discussion in Section 3 contains theorems about the conjectures. Finally, in Section 4 we describe our methods, explaining our experimentation and giving a brief guide to our data, all of which and much more is tabulated and available online at `www.math.tamu.edu/~sottile/pages/Flags/`. We also describe some interesting phenomena we observed in our data.

1. Background

1.1. **Basics on flag manifolds.** Given positive integers $a := (a_1 < \cdots < a_k)$ with $a_k < n$, let $\mathbb{F}\ell(a;n)$ be the manifold of flags in $\mathbb{C}^n$ of *type* a,

$$\mathbb{F}\ell(a;n) \;:=\; \{E_\bullet = E_{a_1} \subset E_{a_2} \subset \cdots \subset E_{a_k} \subset \mathbb{C}^n \mid \dim E_{a_i} = a_i\}\,.$$

If we set $a_0 := 0$, then this has dimension $\dim(a) := \sum_{i=1}^{k}(n-a_i)(a_i - a_{i-1})$. Flags $F_\bullet$ of type $1 < 2 < \cdots < n-1$ in $\mathbb{C}^n$ are called *complete*.

The positions of flags $E_\bullet$ of type a relative to a fixed complete flag $F_\bullet$ stratify $\mathbb{F}\ell(a;n)$ into *Schubert cells* whose closures are *Schubert varieties.* These have a precise description in terms of linear algebra and combinatorics. Define $W^a \subset S_n$ to be the set of permutations of $\{1,2,\dots,n\}$ with descents in a,

$$W^a \;:=\; \{w \in S_n \mid i \notin \{a_1,\dots,a_k\} \Rightarrow w(i) < w(i+1)\}\,.$$

Permutations $w \in W^a$ index Schubert cells $X_w^\circ F_\bullet$ and Schubert varieties $X_w F_\bullet$. Precisely, if we set $r_w(i,j) = |\{l \le i \mid j + w(l) > n\}|$, then

$$\begin{aligned} X_w^\circ F_\bullet \;&=\; \{E_\bullet \mid \dim E_{a_i} \cap F_j = r_w(a_i,j),\; i=1,\dots,k,\; j=1,\dots,n\}, \quad \text{and}\\ X_w F_\bullet \;&=\; \{E_\bullet \mid \dim E_{a_i} \cap F_j \ge r_w(a_i,j),\; i=1,\dots,k,\; j=1,\dots,n\}\,. \end{aligned} \tag{1.1}$$

Flags $E_\bullet$ in $X_w^\circ F_\bullet$ have *position w relative to $F_\bullet$*. We call a permutation $w \in W^a$ a *Schubert condition* on flags of type a. These irreducible subvarieties have codimension in $\mathbb{F}\ell(a;n)$ equal to the length, $\ell(w)$, of the index w.

We have $X_w^\circ F_\bullet \simeq \mathbb{C}^{|a|-\ell(w)}$. We use a convenient set of coordinates for the Schubert cells. Let $\mathcal{M}_w$ be the set of $a_k \times n$-matrices M whose entries $x_{i,j}$ satisfy

$$\begin{aligned} x_{i,w(i)} \;&=\; 1\\ x_{i,j} \;&=\; 0 \qquad \text{if } j < w(i) \text{ or } w^{-1}(j) < i\\ &\qquad\qquad \text{or if } a_l < i < w^{-1}(j) \le a_{l+1}, \text{ for some } l\,, \end{aligned}$$

and whose remaining $\dim(a) - \ell(w)$ entries give coordinates for $\mathcal{M}_w$. For example, if $n=8$, $a=(2,3,6)$, and $w = 25\,3\,167\,48$, then $\mathcal{M}_w$ consists of matrices of the form

$$\begin{pmatrix} 0 & 1 & x_{13} & x_{14} & 0 & x_{16} & x_{17} & x_{18}\\ 0 & 0 & 0 & 0 & 1 & x_{26} & x_{27} & x_{28}\\ 0 & 0 & 1 & x_{34} & 0 & x_{36} & x_{37} & x_{38}\\ 1 & 0 & 0 & x_{44} & 0 & 0 & 0 & x_{48}\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x_{58}\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_{68} \end{pmatrix}.$$

The relation of $\mathcal{M}_w$ to the Schubert cell is as follows. Fix a complete flag $F_\bullet$, and choose an ordered basis $e_1,\dots,e_n$ for $\mathbb{C}^n$, corresponding to the columns of our matrices, such that F_i is the linear span of the last i basis vectors, $e_{n+1-i},\dots,e_{n-1},e_n$. Given a matrix $M \in \mathcal{M}_w$, set E_{a_i} to be the row space of the first a_i rows of M. Then the flag $E_\bullet$ has type a and lies in the Schubert cell $X_w^\circ F_\bullet$, every flag $E_\bullet \in X_w^\circ F_\bullet$ arises in this way, and the association $M \mapsto E_\bullet$ is an algebraic bijection between $\mathcal{M}_w$ and $X_w^\circ F_\bullet$.

Let ι be the identity permutation. Then $\mathcal{M}_\iota$ provides local coordinates for $\mathbb{F}\ell(a;n)$ in which a Schubert variety $X_w F_\bullet$ has nice equations. Note that

$$\dim(E_{a_i} \cap F_j) \;\ge\; r \quad\iff\quad \operatorname{rank}(A) \;\le\; a_i + j - r\,,$$

where A is the matrix formed by stacking the first a_i rows of $\mathcal{M}_\iota$ on top of a $j \times n$ matrix for F_j. This rank condition is the vanishing of all minors of A of size $1+a_i+j-r$.

When $b = (b)$ is a singleton, $\mathbb{F}\ell(b;n)$ is the Grassmannian of b-planes in $\mathbb{C}^n$, written $G(b,n)$. Non-identity permutations in W^b have a unique descent at b and are called

Grassmannian. Our conjecture concerns *Grassmannian Schubert varieties*, which are Schubert varieties of any flag manifold indexed by Grassmannian permutations.

1.2. **The Shapiro Conjecture.** A list $(w_1,\ldots,w_m)$ of permutations in W^a is called *Schubert data* if $\ell(w_1)+\cdots+\ell(w_m)=\dim(a)$. Schubert data index enumerative geometric problems involving Schubert varieties. Specifically, given such Schubert data and complete flags $F_\bullet{}^1,\ldots,F_\bullet{}^m$, consider the Schubert intersection

$$X_{w_1}F_\bullet{}^1\cap\cdots\cap X_{w_m}F_\bullet{}^m\,. \tag{1.2}$$

When the flags are in general position, this intersection is zero-dimensional, and it equals the intersection of corresponding Schubert cells. In that case, the intersection (1.2) consists of the flags $E_\bullet$ of type a which have position w_i relative to $F_\bullet{}^i$, for each $i=1,\ldots,m$. Counting the solutions is a problem in enumerative geometry.

The degree of a zero-dimensional intersection (1.2) does not depend on the choice of flags and we call this number $d(w_1,\ldots,w_m)$ the *degree* of the Schubert data. When the intersection is transverse, this degree counts the solutions to the problem of enumerating flags of type a having positions $w_1,\ldots,w_m$ relative to the fixed chosen flags. Transversality is guaranteed if the chosen flags are in general position [10].

The Shapiro conjecture concerns the following variant to this classical enumerative geometric problem: What *real* flags $E_\bullet$ have given position w_i relative to *real* flags $F_\bullet{}^i$, for each $i=1,\ldots,m$? In the Shapiro conjecture, the flags $F_\bullet{}^i$ are not taken to be general real flags, but rather flags osculating a rational normal curve. Let $\gamma\colon\mathbb{C}\to\mathbb{C}^n$ be the rational normal curve, $\gamma\colon t\mapsto(t^{n-1},t^{n-2},\ldots,t,1)$. The *osculating flag* $F_\bullet(t)$ of subspaces to the rational normal curve at the point $\gamma(t)$ is

$$F_\bullet(t)\ :=\ \operatorname{span}\{\gamma(t),\frac{d\gamma}{ds}(t),\ldots,\frac{d^{i-1}\gamma}{ds^{i-1}}(t)\}\,.$$

When $t=0$, the flag $F_\bullet(0)$ is the standard flag we used in the description of $\mathcal{M}_w$.

Conjecture 1.3 (B. Shapiro and M. Shapiro). *Suppose that $w_1,\ldots,w_m$ is Schubert data for flags of type a. If the flags $F_\bullet{}^1,\ldots,F_\bullet{}^m$ osculate the rational normal curve at distinct real points, then the intersection* 1.2 *is transverse and consists only of real points.*

Write $X_w(t)$ for $X_wF_\bullet(t)$ and X_w for $X_wF_\bullet(0)$. Then the Shapiro conjecture is concerned with intersections of the form

$$X_{w_1}(t_1)\cap X_{w_2}(t_2)\cap\cdots\cap X_{w_m}(t_m)\,. \tag{1.4}$$

Previously, Conjecture 1.3 has mostly been studied for Grassmannians. Experimental evidence for its validity was first found by one of us (Sottile) [13, 17]. This led to a more systematic investigation, both experimentally and theoretically [19]. There, relationships between the conjecture for different collections of Schubert data on different Grassmannians were studied. For example, if the Shapiro conjecture holds for a Grassmannian for the Schubert data consisting only of codimension 1 conditions, then it holds for all Schubert data on that Grassmannian and on all smaller Grassmannians. (Perhaps dropping the claim of transversality.) More recently, Eremenko and Gabrielov proved the conjecture for any Schubert data on a Grassmannian of codimension 2-planes [4]. This result is appealingly interpreted as a rational function with real critical points must be real.

The original conjecture was for flag manifolds. Unfortunately, a counterexample was found in [19]. Subsequent experimentation refined this counterexample, and suggested a reformulation of the original conjecture that might be true. We study this refined conjecture, and report on massive computer experimentation (over 12 GHz-years) studying this conjecture. A by-product of this experimentation was the discovery of several new and unusual phenomena, which we describe through examples. The first is a relatively simple counterexample to the Shapiro conjecture.

1.3. **The Shapiro conjecture is false for flags in space.** Use a^b to indicate that condition a is repeated b times. Then $\big((1324)^3, (1243)^2\big)$ is Schubert data for flags of type $(2,3)$ in $\mathbb{C}^4$. For distinct points $s,t,u,v,w \in \mathbb{RP}^1$, consider the Schubert intersection

$$X_{1324}(s)\cap X_{1324}(t)\cap X_{1324}(u)\ \cap\ X_{1243}(v)\cap X_{1243}(w)\,, \tag{1.5}$$

in projective 3-space. There, a partial flag of type $(2,3)$ consists of a line ℓ lying on a plane H. Then $(\ell\subset H)\in X_{1243}(v)$ if the plane H contains the point $\gamma(v)$, and $(\ell\subset H)\in X_{1324}(s)$ if ℓ meets the tangent line $\ell(s)$ to γ at $\gamma(s)$.

Suppose that the flag $\ell\subset H$ lies in the intersection (1.5). Then H contains the two points $\gamma(v)$ and $\gamma(w)$, and hence the secant line $\lambda(v,w)$ that they span. Since ℓ is another line in H, ℓ meets this secant line $\lambda(v,w)$. As long as $\ell\neq\lambda(v,w)$, then ℓ determines H uniquely as the span of ℓ and $\lambda(v,w)$. In this way, we are reduced to determining the lines ℓ which meet the three tangent lines $\ell(s)$, $\ell(t)$, $\ell(u)$, and the secant line $\lambda(v,w)$.

The set of lines which meet the three mutually skew lines $\ell(s)$, $\ell(t)$, and $\ell(u)$ forms one ruling of a quadric surface Q in $\mathbb{P}^3$. We display this quadric Q and the ruling in Figure 1, as well as the rational normal curve with its three tangent lines. The lines meeting these

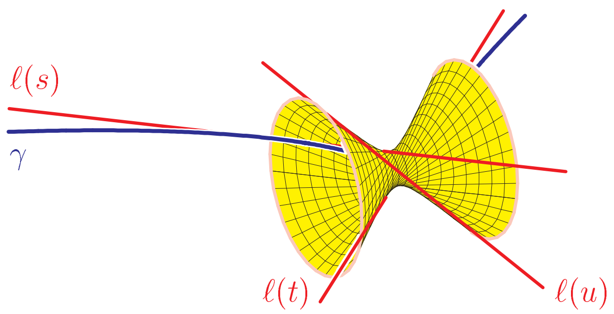

FIGURE 1. Quadric containing three lines tangent to the rational normal curve

three and the secant line $\lambda(v,w)$ correspond to the points where $\lambda(v,w)$ meets the quadric Q. In Figure 2, we display a secant line which meets the hyperboloid in two points, and therefore these choices give two real flags in the intersection (1.5). There is also a secant line which meets the hyperboloid in two complex conjugate points. For this secant line, the two flags in the intersection (1.5) are both complex. We show this in Figure 3.

Since any three points of $\mathbb{RP}^1$ may be carried to any other three points via a real Möbius transformations, *and* these extend to projective automorphisms of $\mathbb{P}^3$, we have

Proposition 1.6. *The intersection* (1.5) *is transversal and consists only of real points if there are disjoint intervals* I_2 *and* I_3 *of* $\mathbb{RP}^1$ *so that* $s,t,u\in I_2$ *and* $v,w\in I_3$.

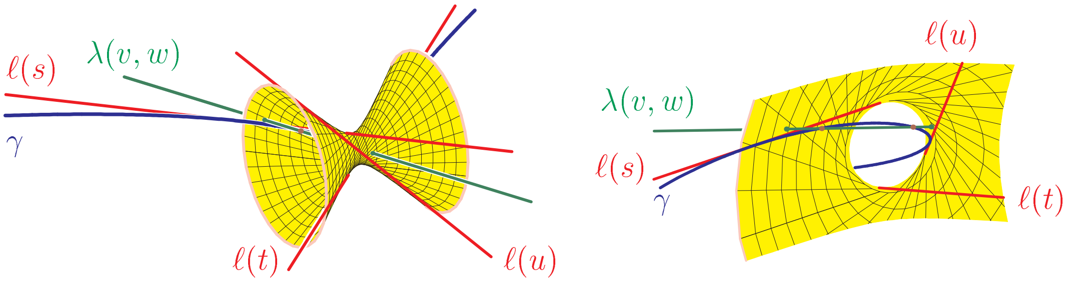

FIGURE 2. Two views of secant line meeting Q

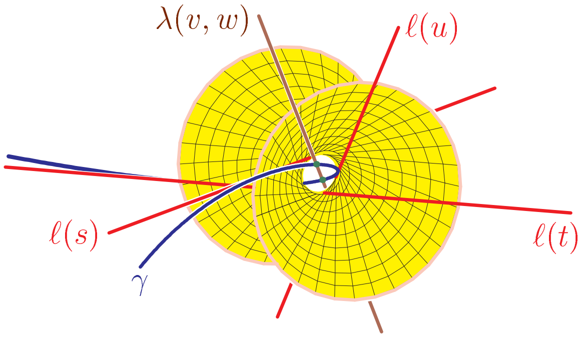

FIGURE 3. Secant line not meeting Q

Proposition 1.6 suggests a refinement to the Shapiro conjecture which may hold. We describe that refinement and experimental evidence in its favor.

2. RESULTS

Experimentation designed to test hypotheses is a primary means of inquiry in the natural sciences. In mathematics we typically use proof and example as our primary means of inquiry. Many mathematicians (including the authors) feel that they are striving to understand the nature of objects that inhabit a very real mathematical reality. For us, experimentation plays an important role in helping to formulate conjectures, which are then studied and perhaps eventually decided. We discuss the conjectures which were informed by our massive experimentation that is described in Section 4.

2.1. **Conjectures.** Let $a = (a_1 < \cdots < a_k)$ and n be positive integers with $a_k < n$. A permutation $w \in W^a$ is *Grassmannian* if it has a single descent, say at position a_l. Then the Schubert variety $X_w F_\bullet$ of $\mathbb{F}\ell(a; n)$ is the pull-back of the Schubert variety $X_w F_\bullet$ of the Grassmannian $G(a_l, n)$. Write $\delta(w)$ for the unique descent of w.

Suppose that $w_1, \ldots, w_m \in W^a$ are data for a Schubert problem on $\mathbb{F}\ell(a; n)$ with each permutation w_i Grassmannian. A collection of points $t_1, \ldots, t_m \in \mathbb{RP}^1$ is *monotone* (with respect to $w_1, \ldots, w_m$) if the function

$$t_i \longmapsto \delta(w_i) \in \{a_1, a_2, \ldots, a_k\}$$

is monotone, when the ordering of the t_i is consistent with some orientation of $\mathbb{RP}^1$.

Conjecture 2.1. *Suppose that* $w_1, \ldots, w_m \in W^a$ *are data for a Schubert problem where each permutation* w_i *is Grassmannian. Then the intersection*

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,, \tag{2.2}$$

is transverse with all points of intersection real, if the points $t_1, \ldots, t_m \in \mathbb{RP}^1$ *are monotone with respect to* $w_1, \ldots, w_m$.

Note that if $\mathbb{F}\ell(a;n)$ is a Grassmannian, then every choice of points is monotone, so the Shapiro conjecture for Grassmannians is a special case of Conjecture 2.1.

Our experimentation systematically investigated the original Shapiro conjecture for flag manifolds, with a focus on this monotone conjecture. In all, we examined 591 such Grassmannian Schubert problems on 29 different flag manifolds. On these, we verified that on each of 140 million specific monotone intersections of the form (2.2) had all solutions real. We find this to be overwhelming evidence in support of our monotone conjecture.

Remark 2.3. Conjecture A of the Introduction is the instance of Conjecture 2.1 for the flag manifold $\mathbb{F}\ell(2<3;5)$ for the Schubert data $\big((13245)^4,\,(12435)^4\big)$, expressed in terms of polynomial systems in local coordinates.

Remark 2.4. The example of Section 1.3 illustrates both Conjecture 2.1 and its limitations. The condition on disjoint intervals I_2 and I_3 of Proposition 1.6 is equivalent to the monotone choice of points in Conjecture 2.1. Also note that the choices which give no real solutions are not monotone.

We give a stronger conjecture which is supported by our experimental investigation. It ignores the issue of reality and concentrates only on the transversality of an intersection.

Conjecture 2.5. *If* $w_1, \ldots, w_m \in W^a$ *are data for a Schubert problem where each permutation* w_i *is Grassmannian then the intersection*

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,,$$

is transverse, if the points $t_1, \ldots, t_m \in \mathbb{RP}^1$ *are monotone with respect to* $w_1, \ldots, w_m$.

Theorem 2.6. *Conjecture* 2.5 *implies Conjecture* 2.1.

We prove this in Section 3. In our experimentation, we kept track of which polynomial systems corresponded to non-transverse intersections. No such polynomial system came from a monotone choice.

Eremenko, Gabrielov, Shapiro, and Vainshtein recently established Conjecture 2.1 for manifolds of flags of type $(n-2, n-1)$ in $\mathbb{C}^n$ [5].

3. Discussion

For $m \leq n - a_i$, the *special* Schubert variety in $G(a_i, n)$ is defined by

$$\Omega_{(m)}F_\bullet \;:=\; \{E_{a_i} | \dim\, E_{a_i} \cap F_m \geq 1\}\,.$$

The inverse image under the projection map $\pi_{a_i} : \mathbb{F}\ell(a;n) \to G(a_i,n)$ sending $E_\bullet \mapsto E_{a_i}$ is a Schubert variety of $\mathbb{F}\ell(a;n)$ defined by the same formula and we use $\Omega_{(m)}F_\bullet$ for this subvariety as well. When $F_\bullet = F_\bullet(t)$ osculates the rational normal curve at a point $t \in \mathbb{P}^1$, we denote the corresponding Schubert variety by $\Omega_{(m)}(t) := \Omega_{(m)}F_\bullet(t)$.

W^{a_i} coincides with the set $\binom{[n]}{a_i}$ of subsets of $\{1,\ldots,n\}$ of size a_i. For each $i=1,\ldots,k$, define a map $\pi_{a_i}: W^a \to \binom{[n]}{a_i}$ by $\pi_{a_i}(v) := \{v_1,\ldots,v_{a_i}\}$.

Define the *a_i-Bruhat order* on W^a by its covers: if $u \lessdot v$ is a cover in the Bruhat order, then $u \lessdot_{a_i} v$ if and only if $\pi_{a_i}(u) < \pi_{a_i}(v)$. Equivalently, $u \lessdot_{a_i} v$ if and only if $v = u\sigma_{bc}$, where σ_{bc} is the transposition interchanging $b \leq a_i < c$. Note in particular that $u <_{a_i} v$ implies $\pi_{a_i}(u) < \pi_{a_i}(v)$. Define a set $\mathcal{L}_{a_i,m}(w) \subset W^a$ by

$$\mathcal{L}_{a_i,m}(w) \quad := \quad \{v \in W^a \mid w <_{a_i} v,\ \ell(v) = \ell(w)+m,\ \#\{l > a_i | v(l) \neq w(l)\} = m\}.$$

The following result is established in [16].

Theorem 3.1. *In the Chow ring, $A^*(\mathbb{F}\ell(a;n))$, the cycle-theoretic equality holds:*

$$[X_w F_\bullet] \cdot [\Omega_{(m)}(t)] = \sum_{v \in \mathcal{L}_{a_i,m}(w)} [X_v F_\bullet]$$

Eisenbud and Harris [1, Theorem 8.1] proved that the scheme-theoretic limit $\lim_{s\to 0} X_w F_\bullet \cap \Omega_{(m)}(t)$ is supported on a union of Schubert varieties, when the *flag manifold is a Grassmannian.* We remove this restriction to Grassmannians.

Theorem 3.2. *The equality*

$$\lim_{t\to 0} X_w F_\bullet \cap \Omega_{(m)}(t) = \sum_{v \in \mathcal{L}_{a_i,m}(w)} X_v F_\bullet \tag{3.3}$$

holds scheme-theoretically.

Our proof follows that of [1]. This was proven when $m=1$ in [20]. It has a number of consequences for the monotone conjecture.

Theorem 3.4. *If Conjecture 2.1 holds for all Schubert data on a flag manifold $\mathbb{F}\ell(a;n)$ involving codimension 1 conditions (each permutation w_i has length 1), then it holds for all Schubert data on $\mathbb{F}\ell(a;n)$.*

Proof of Theorem 2.6. If Conjecture 2.5 holds, then by [20, Corollary 2.2], Conjecture 2.1 holds for all Schubert intersection problems (2.2) involving only codimension 1 Schubert conditions. By Theorem 3.4, this implies Conjecture 2.1 for all Schubert intersection problems. □

4. Methods

At the heart of our experiments is the computation the number of real roots of a real polynomial system. This system generates the ideal of the Schubert intersection

$$X_{w_1}(t_1) \cap X_{w_2}(t_2) \cap \cdots \cap X_{w_m}(t_m)\,,$$

which depends upon the points $t_1,\ldots,t_m \in \mathbb{RP}^1$ at which the fixed flags osculate the curve γ. Our computational procedure takes the following data as input:

(1) A compact description of the Schubert problem.
(2) The number of complex solutions to the Schubert problem.
(3) A list of *necklaces*, that is, the combinatorial types of the order along the rational normal curve at which the conditions will be evaluated.

(4) A list, $L \subset \mathbb{RP}^1$, of points where the conditions are to be evaluated.

These data are used to create a Singular [6] input file that controls the first stage of the computation. In this stage, a subset of the points L is selected randomly, then each necklace determines which Schubert conditions are evaluated at which osculating flags. Each such choice is one instance of the Schubert problem, whose ideal is written in the Singular input file. Singular is called, and it uses Gröbner bases to compute an eliminant (a univariate polynomial in the ideal of the Schubert intersection) for each instance. Finally, Maple is called to compute the number of real roots of each eliminant using the package `realroot`, and a table (described below) is updated accordingly. Instances for which an eliminant could not be computed were set aside and studied by hand.

One iteration of the procedure just described yields one instance of the Schubert problem for each necklace. The complete computation is organized by a shell script which iterates this procedure a fixed number of times (typically several hundred to several tens of thousands), randomly selecting a new set of points L each time.

Table 1 shows the results of computing 800,000 instances of the Schubert problem

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
22223333	0	0	0	0	0	0	100000
22322333	0	0	21	16129	33686	29350	20814
22233233	0	0	31	16276	33430	29194	21069
22332233	0	0	421	13742	46961	23561	15315
22323323	0	0	6242	22480	36329	26522	8427
22332323	0	504	18532	27844	30962	15546	6612
22232333	0	1846	8414	13887	25079	20784	29990
23232323	3830	10131	32326	21679	20790	8066	3178

TABLE 1. Table for the problem $13245^4 \cdot 12435^4 = 12$ on $\mathbb{F}\ell(2,3;5)$

$(13245^4, 12435^4)$ on $\mathbb{F}\ell(2 < 3; 5)$. Each row records the number of times a given number of real solutions was observed for a given necklace. The entries in the first column represent the necklaces as sequences $\{\delta(w_1), \ldots, \delta(w_m)\}$, where $\delta(w)$ denotes the unique descent of the Grassmannian permutation w, as described in Section 2.1. Thus, in this case, a 2 represents the condition on the 2 plane E_2 given by the permutation 1324, and a 3 represents the condition on E_3 given by the permutation 1243. The necklace corresponding to the first row is monotone.

4.1. **Observations.** This experimentation not only studied Conjecture 2.1, but it systematically studied the original Shapiro conjecture. Many interesting phenomena were observed. For example, there is an extension of Conjecture 2.1 in which some conditions are not Grassmannian (and a similar extension of Theorem 3.4). We do not state it here in the extended abstract.

We found many Schubert problems with an apparent *lower bound* on their number of real solutions. For example, Table 2 is from the problem on $\mathbb{F}\ell(2, 3; 6)$ with Schubert data

$(132456^5, 125346^3)$, which has degree 14. Such lower bounds on the number of real solu-

Necklace	Number of Real Solutions							
	0	2	4	6	8	10	12	14
22222333	0	0	0	0	0	0	0	3000
22223233	0	0	0	0	27	562	887	1524
22232323	0	0	4	77	474	776	810	859
22232233	0	0	34	69	331	915	1063	588
22322323	0	0	152	356	634	839	726	293

TABLE 2. Table for the problem $132456^5 \cdot 125346^3 = 14$ on $\mathbb{F}\ell(2,3;6)$

tions to enumerative geometric problems were first found by Eremenko and Gabrielov [3] in the context of the Shapiro conjecture for Grassmannians. Lower bounds have also been observed for rational curves on surfaces [8, 11, 24] and for sparse polynomial systems [15].

The problem $(312564^2, 124356^5)$ on $\mathbb{F}\ell(1,3,5;6)$ has degree 10, and the condition 312564 is *not* Grassmannian and 124356 is Grassmannian with descent at 3. Not only does this problem exhibit a lower bound, but it also has apparent 'gaps' in the possible numbers of real solutions. Table 3 gives the data from this computation. In each necklace, A represents the condition 312564, while 3 represents the Grassmannian condition 124356. This is a new phenomena first observed in some sparse polynomial systems [15].

Necklace	Number of Real Solutions					
	0	2	4	6	8	10
$A33A333$	0	1850	0	10381	0	7769
$A3A3333$	0	3177	0	13729	0	3094
$AA33333$	0	11222	0	8397	0	381

TABLE 3. Table for the problem $312564^2 \cdot 124356^5 = 10$ on $\mathbb{F}\ell(1,3,5;6)$

One unusual problem we looked at was on the flag manifold $\mathbb{F}\ell(2,4;6)$ and it involved four identical, but non-Grassmannian conditions, 142536. It always had real solutions, and by using some geometric reasoning we can prove that is always the case.

Theorem 4.1. *For any distinct $s,t,u,v \in \mathbb{RP}^1$, then intersection*

$$X_{142536}(s) \cap X_{142536}(t) \cap X_{142536}(u) \cap X_{142536}(v)$$

is transverse and consists of 6 *real points.*

Another interesting feature of this problem is that its Galois group [7] is not the full symmetric group S_6, but rather the symmetric group S_3. This is the smallest known example of an enumerative geometric problem in the Schubert calculus whose Galois group is not the full symmetric group, and it is strikingly small.

A final phenomena that we discovered is a Schubert problem which is not transverse, when it involves flags osculating a rational normal curve. This may have negative repercussions for part of Varchenko's program concerning the Bethe Ansatz and Fuchsian equations [12]. It was quite unexpected, as Eisenbud and Harris showed that on a Grassmannian, any intersection

$$X_{w_1}(s_1) \cap X_{w_2}(s_2) \cap \cdots \cap X_{w_m}(s_m) \tag{4.2}$$

has the expected dimension $\dim(a) - \sum \ell(w_i)$, if the points $s_1, s_2, \ldots, s_m \in \mathbb{P}^1$ are distinct [1, Theorem 2.3]. Our example shows that the result of Eisenbud and Harris cannot be extended to the flag manifold.

The manifold of flags of type $(1,3)$ in $\mathbb{C}^5$ has dimension 8. Since $\ell(32514) = 5$ and $\ell(21435) = 2$, there are no flags of type $(1,3)$ satisfying the trio of Schubert conditions (32514, 21435, 21435) imposed by three general flags. If however the flags osculate a rational normal curve γ, then the intersection is nonempty.

Theorem 4.3. *$X_{32514}(u) \cap X_{21435}(s) \cap X_{21435}(t) \neq \emptyset$ for all $s, t, u \in \mathbb{P}^1$.*

Proof. We may assume that $u = 0$, so that flags in $X^\circ_{32514}(u)$ are given by matrices in $\mathcal{M}_{32514}$. Consider the 3×5 matrix in $\mathcal{M}_{32514}$ with row vectors v_1, v_2, and v_3:

$$\begin{bmatrix} 0 & 0 & 1 & \frac{3}{2}(s+t) & 6st \\ 0 & 1 & 0 & -3st & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $E_\bullet \colon E_1 \subset E_3$ be the corresponding flag, which lies in $X^\circ_{32514}(\infty)$. Define

$$\lambda(s) \;:=\; (s^4,\, -4s^3,\, 6s^2,\, -4s,\, 1) \;\in\; (\mathbb{C}^5)^*\,,$$

and note that $\lambda(s)$ annihilates $\gamma(s)$, $\gamma'(s)$, $\gamma''(s)$, and $\gamma'''(s)$, where $\gamma(s) := (1, s, s^2, s^3, s^4)$ parametrizes the rational normal curve in a neighborhood of ∞. Thus $\lambda(s)$ is the linear form annihilating the 4-plane $F_4(s)$ osculating the rational normal curve γ at the point $\gamma(s)$. Since $v_1 \cdot \lambda(s)^t = 0$, we have $E_1 \subset F_4(s)$. Finally,

$$2sv_1 + v_2 + (4s^3 - 6st)v_3 \;=\; (0,\, 1,\, 2s,\, 3s^2,\, 4s^3) \;=\; \gamma'(s)\,,$$

so $E_3 \cap F_2(s) \neq 0$ and thus $E_\bullet \in X_{21435}(s)$. Similarly, $E_\bullet \in X_{21435}(t)$. $\square$

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