# STRUCTURE OF THE MALVENUTO-REUTENAUER HOPF ALGEBRA OF PERMUTATIONS (EXTENDED ABSTRACT)

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Abstract. We analyze the structure of the Malvenuto-Reutenauer Hopf algebra of permutations in detail. We give explicit formulas for its antipode, prove that it is a cofree coalgebra, determine its primitive elements and its coradical filtration and show that it decomposes as a crossed product over the Hopf algebra of quasi-symmetric functions. We also describe the structure constants of the multiplication as a certain number of facets of the permutahedron. Our results reveal a close relationship between the structure of this Hopf algebra and the weak order on the symmetric groups.

Résumé. On analyse la structure de l'algèbre de Hopf de Malvenuto et Reutenauer en détail. On donne des formules explicites pour son antipode, on prouve que c'est une coalgèbre colibre, on determine ses éléments primitifs et sa filtration coradical et on montre qu'elle se décompose comme un produit croisé sur l'algèbre de Hopf de fonctions quasi-symétriques. On decrit aussi les constants de structure de la multiplication comme un certain numéro de facettes du permutahedron. Nos résultats mettent en évidence une forte relation entre la structure de cette algèbre de Hopf et l'ordre faible dans les groupes symetriques.

## Introduction

Malvenuto [15] introduced the Hopf algebra $\mathfrak{S}Sym$ of permutations, which has a linear basis $\{\mathcal{F}_u \mid u \in \mathfrak{S}_n, n \geq 0\}$ indexed by permutations in all symmetric groups $\mathfrak{S}_n$. The Hopf algebra $\mathfrak{S}Sym$ is non-commutative, non-cocommutative, self-dual, and graded. Among its sub- and quotient- Hopf algebras are many algebras central to algebraic combinatorics. These include the algebra of symmetric functions [14, 22], Gessel's algebra $\mathcal{Q}Sym$ of quasi-symmetric functions [11], the algebra of non-commutative symmetric functions [10], the Loday-Ronco algebra of planar binary trees [12], Stembridge's algebra of peaks [23], the Billera-Liu algebra of Eulerian enumeration [4], and others. The structure of these combinatorial Hopf algebras with respect to certain distinguished bases has been an important theme in algebraic combinatorics, with applications to the combinatorial problems these algebras were created to study. We give a detailed understanding of the structure of $\mathfrak{S}Sym$, both in algebraic and in combinatorial terms.

Our main tool is a new basis $\{\mathcal{M}_u \mid u \in \mathfrak{S}_n,\ n \geq 0\}$ for $\mathfrak{S}Sym$ related to its original basis by Möbius inversion on the weak order of the symmetric groups. These bases $\{\mathcal{M}_u\}$

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and $\{\mathcal{F}_u\}$ are analogous to the monomial and fundamental basis of $\mathcal{Q}Sym$, which are related via Möbius inversion on their index sets, the Boolean posets $\mathcal{Q}_n$.

We give enumerative-combinatorial descriptions of the product, coproduct, and antipode of $\mathfrak{S}Sym$ with respect to the basis $\{\mathcal{M}_u\}$. For example, the coproduct is obtained by splitting a permutation at certain special positions that we call global descents. Descents and global descents are left adjoint and right adjoint to a natural map $\mathcal{Q}_n \to \mathfrak{S}_n$.

The structure constants for the product with respect to the basis $\{\mathcal{M}_u\}$ are nonnegative integers with the following geometric-combinatorial description. The 1-skeleton of the permutahedron $\Pi_{n-1}$ is the Hasse diagram of the weak order on $\mathfrak{S}_n$. The facets of the permutahedron are canonically isomorphic to products of lower dimensional permutahedra. Say that a facet isomorphic to $\Pi_{p-1} \times \Pi_{q-1}$ has type $(p, q)$. Given $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, such a facet has a distinguished vertex corresponding to $(u, v)$ under the canonical isomorphism. Then, for $w \in \mathfrak{S}_{p+q}$, the coefficient of $\mathcal{M}_w$ in the product $\mathcal{M}_u \cdot \mathcal{M}_v$ is the number of facets of the permutahedron $\Pi_{p+q-1}$ of type $(p, q)$ with the property that the distinguished vertex is below $w$ and closer to $w$ than to any other vertex in the facet.

We also give explicit formulas for the antipode with respect to both bases. The structure constants with respect to the basis $\{\mathcal{M}_u\}$ have constant sign, as in the case of $\mathcal{Q}Sym$. The situation is more complicated for the basis $\{\mathcal{F}_u\}$, which may explain why no such explicit formulas were previously known.

Elucidating the elementary structure of $\mathfrak{S}Sym$ with respect to the basis reveals further algebraic structures. For example, $\mathfrak{S}Sym$ is a cofree graded coalgebra. A consequence is that the coradical filtration of $\mathfrak{S}Sym$ (which encapsulates the complexity of iterated coproducts) is the algebraic counterpart of a filtration of the symmetric groups by certain lower order ideals. In particular, the space of primitive elements is spanned by the set $\{\mathcal{M}_u \mid u \text{ has no global descents}\}$. Cofreenes was shown by Poirier and Reutenauer [20], in dual form, through the introduction of a different basis. The study of primitive elements was pursued from this point of view by Duchamp, Hivert, and Thibon [7].

There is a well-known morphism of Hopf algebras $\mathfrak{S}Sym \to \mathcal{Q}Sym$ that maps one fundamental basis onto the other, by associating to a permutation $u$ its descent set $\mathrm{Des}(u)$. We describe this map in terms of the bases $\{\mathcal{M}_u\}$ and $\{M_\alpha\}$.

Lastly, $\mathfrak{S}Sym$ decomposes as a crossed product over $\mathcal{Q}Sym$. This construction from the theory of Hopf algebras is a generalization of the notion of group extensions. We provide a combinatorial description for the Hopf kernel of the map $\mathfrak{S}Sym \to \mathcal{Q}Sym$.

These results are expanded on and proven in the manuscript [2] of the same name. For a background on quasi-symmetric functions, see [22, §7.19], for Hopf Algebras, we recommend the book of Montgomery [18]. We also recommend the papers [20] of Poirier and Reutenauer and of [7] of Duchamp, Thibon, and Hivert, who studied this same Hopf algebra of permutations from a different perspective, the latter under the name 'free quasi-symmetric functions'.

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## 1. Essential definitions

1.1. **Quasi-symmetric functions.** Gessel [11] introduced the algebra $\mathcal{Q}Sym$ of quasi-symmetric functions as the natural target for Stanley's $P$-partition generating function. Subsequent work has shown its centrality, even universality, for generating functions in algebraic combinatorics [9, 3, 1].

A sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers is a *composition of* $n$ if $\sum_i \alpha_i = n$. Compositions of $n$ correspond to subsets of $[n-1]$ as follows

$$\alpha \;=\; (\alpha_1, \alpha_2, \ldots, \alpha_k) \quad\longleftrightarrow\quad I(\alpha) \;=\; \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\}\ .$$

These subsets of $[n-1]$ (and thus compositions of $n$) form the Boolean poset $\mathcal{Q}_n$, and the induced order relation on compositions is called *refinement*.

For $\mathsf{S} \subseteq [n-1]$, the *fundamental quasi-symmetric* function $F_{\mathsf{S},n}$ is

$$F_{\mathsf{S},n} \;:=\; \sum_{\substack{j_1 \leq \cdots \leq j_n \\ i \in \mathsf{S} \Rightarrow j_i < j_{i+1}}} x_{j_1} x_{j_2} \ldots x_{j_n}\ .$$

These form a basis for $\mathcal{Q}Sym$. Another basis is provided by the monomial quasi-symmetric functions $M_\alpha$, which are indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$

$$M_\alpha \;:=\; \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}\ .$$

These two bases are related via Möbius inversion on the Boolean poset $\mathcal{Q}_n$.

$$F_\alpha \;=\; \sum_{\alpha \leq \beta} M_\beta \qquad\text{and}\qquad M_\alpha \;=\; \sum_{\alpha \leq \beta} (-1)^{c(\beta)-c(\alpha)} F_\beta\,,$$

where $c((\alpha_1, \ldots, \alpha_k)) = k$ and $(-1)^{c(\beta)-c(\alpha)}$ is the Möbius function of $\mathcal{Q}_n$.

The product of these $M_\alpha$ is given by quasi-shuffles of their indices [9, Lemma 3.3]. A *quasi-shuffle* of compositions $\alpha$ and $\beta$ is a shuffle of the components of $\alpha$ and $\beta$, where in addition we may replace any number of pairs of consecutive components $(\alpha_i, \beta_j)$ in the shuffle by $\alpha_i + \beta_j$. Then we have

$$M_\alpha \cdot M_\beta \;=\; \sum_\gamma M_\gamma\,, \tag{1.1}$$

where the sum is over all quasi-shuffles $\gamma$ of the compositions $\alpha$ and $\beta$. For instance,

$$M_{(2)} \cdot M_{(1,1)} \;=\; M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)} + M_{(1,3)} + M_{(3,1)}\,. \tag{1.2}$$

The unit element $1 = M_{(\ )}$ is indexed by the empty composition.

Let $X$ and $Y$ be ordered alphabets with $X < Y$ their disjoint union ordered as indicated. Substitution $f(X) \mapsto f(X < Y)$ induces a coproduct $\Delta\colon \mathcal{Q}Sym \to \mathcal{Q}Sym \otimes \mathcal{Q}Sym$ whose action on a monomial function is as follows.

$$\Delta\bigl(M_{(\alpha_1,\ldots,\alpha_k)}\bigr) \;=\; \sum_{p=0}^{k} M_{(\alpha_1,\ldots,\alpha_p)} \otimes M_{(\alpha_{p+1},\ldots,\alpha_k)}\,. \tag{1.3}$$

For instance, $\Delta(M_{(2,1)}) = 1 \otimes M_{(2,1)} + M_{(2)} \otimes M_{(1)} + M_{(2,1)} \otimes 1$.

These definitions give $\mathcal{Q}Sym$ the structure of a graded, connected Hopf algebra. The degree $n$ component is spanned by those $M_\alpha$ where $\alpha$ is a composition of $n$. It is connected, as its degree 0 component is 1-dimensional, and it is a Hopf algebra. An explicit formula for the antipode was given by Malvenuto [15, corollaire 4.20] and Ehrenborg [9, Proposition 3.4]

$$S(M_\alpha) \;=\; (-1)^{c(\alpha)} \sum_{\beta \leq \alpha} M_{\widetilde{\beta}}\,. \tag{1.4}$$

Here, if $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$ then $\widetilde{\beta}$ is $\beta$ written in reverse order $(\beta_t, \ldots, \beta_2, \beta_1)$.

1.2. **The Hopf algebra of permutations.** Let $\mathfrak{S}Sym$ be the graded $\mathbb{Q}$-vector space with *fundamental basis* $\{\mathcal{F}_u \mid u \in \mathfrak{S}_n, n \geq 0\}$, graded by $n$. $\mathfrak{S}Sym$ has a graded Hopf algebra structure first considered in Malvenuto's thesis [15, §5.2] and in her work with Reutenauer [16]. Write 1 for the basis element of degree 0.

The product of two basis elements is obtained by shuffling the corresponding permutations, as in the following example.

$$\mathcal{F}_{12} \cdot \mathcal{F}_{312} \;=\; \mathcal{F}_{12534} + \mathcal{F}_{15234} + \mathcal{F}_{15324} + \mathcal{F}_{15342} + \mathcal{F}_{51234} + \mathcal{F}_{51324} + \mathcal{F}_{51342} + \mathcal{F}_{53124} + \mathcal{F}_{53142} + \mathcal{F}_{53412}\,.$$

More precisely, for $p, q > 0$, set

$$\mathfrak{S}^{(p,q)} \;:=\; \{\zeta \in \mathfrak{S}_{p+q} \mid \zeta \text{ has at most one descent, at position } p\}\,.$$

This is the collection of minimal (in length) representatives of left cosets of the Young or parabolic subgroup $\mathfrak{S}_p \times \mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$, called *Grassmannian permutations*. With these definitions, we describe the product. For $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, set

$$\mathcal{F}_u \cdot \mathcal{F}_v \;=\; \sum_{\zeta \in \mathfrak{S}^{(p,q)}} \mathcal{F}_{(u\times v)\cdot\zeta^{-1}}\,. \tag{1.5}$$

This endows $\mathfrak{S}Sym$ with the structure of a graded algebra with unit 1.

The algebra $\mathfrak{S}Sym$ is also a graded coalgebra with coproduct given by all ways of splitting a permutation. For a sequence $(a_1, \ldots, a_p)$ of distinct integers, let its *standard permutation*† $\mathrm{st}(a_1, \ldots, a_p) \in \mathfrak{S}_p$ be the permutation $u$ defined by

$$u_i < u_j \iff a_i < a_j\,. \tag{1.6}$$

For instance, $\mathrm{st}(625) = 312$. The coproduct $\Delta\colon \mathfrak{S}Sym \to \mathfrak{S}Sym \otimes \mathfrak{S}Sym$ is defined by

$$\Delta(\mathcal{F}_u) \;=\; \sum_{p=0}^{n} \mathcal{F}_{\mathrm{st}(u_1,\ldots,u_p)} \otimes \mathcal{F}_{\mathrm{st}(u_{p+1},\ldots,u_n)}\,, \tag{1.7}$$

when $u \in \mathfrak{S}_n$. For instance, $\Delta(\mathcal{F}_{42531})$ is

$$1 \otimes \mathcal{F}_{42531} + \mathcal{F}_1 \otimes \mathcal{F}_{2431} + \mathcal{F}_{21} \otimes \mathcal{F}_{321} + \mathcal{F}_{213} \otimes \mathcal{F}_{21} + \mathcal{F}_{3142} \otimes \mathcal{F}_1 + \mathcal{F}_{42531} \otimes 1\,.$$

$\mathfrak{S}Sym$ is a graded connected Hopf algebra [15, théorème 5.4].

This Hopf algebra $\mathfrak{S}Sym$ has been an object of recent interest [21, 16, 20, 19, 8, 7, 12, 13]. We remark that sometimes it is the dual Hopf algebra that is considered. To compare results, one may use that $\mathfrak{S}Sym$ is self-dual under the map $\mathcal{F}_u \mapsto \mathcal{F}^*_{u^{-1}}$, where $\mathcal{F}^*_{u^{-1}}$ is the element of the dual basis that is dual to $\mathcal{F}_{u^{-1}}$.

To define the *monomial* basis $\{\mathcal{M}_u\}$ for $\mathfrak{S}Sym$ (in analogy to the basis $\{M_\alpha\}$ of $\mathcal{Q}Sym$), we use the weak order on the symmetric groups $\mathfrak{S}_n$. Let $\ell(n)$ count the inversions $\{i < j \mid u_i > u_j\}$ of a permutation $u$. The *weak order* on $\mathfrak{S}_n$ is defined by

$$u \leq v \iff \exists w \in \mathfrak{S}_n \text{ such that } v \;=\; wu \text{ and } \ell(v) = \ell(w) + \ell(u)\,.$$

The cover relation $u \lessdot v$ occurs precisely when $v$ is obtained from $u$ by transposing a pair of consecutive *values* of $u$; a pair $(u_i, u_j)$ such that $i < j$ and $u_j = u_i + 1$. The maximum element of $\mathfrak{S}_n$ is $\omega_n = (n, \ldots, 2, 1)$. Figure 1 shows the weak order on $\mathfrak{S}_4$.

†Some authors call this flattening.

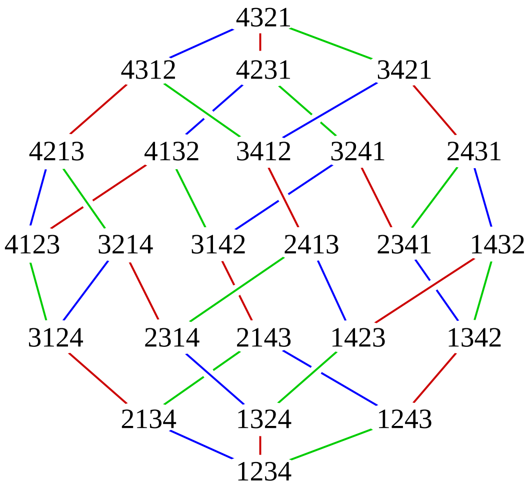


FIGURE 1. The weak order on $\mathfrak{S}_4$

For each $n \geq 0$ and $u \in \mathfrak{S}_n$, define

$$\mathcal{M}_u := \sum_{u \leq v} \mu_{\mathfrak{S}_n}(u, v) \cdot \mathcal{F}_v \,, \tag{1.8}$$

where $\mu_{\mathfrak{S}_n}(\cdot, \cdot)$ is the Möbius function of the weak order in $\mathfrak{S}_n$. By Möbius inversion,

$$\mathcal{F}_u := \sum_{u \leq v} \mathcal{M}_v \,, \tag{1.9}$$

so these elements $\mathcal{M}_u$ indeed form a basis of $\mathfrak{S}Sym$. For instance,

$$\mathcal{M}_{4123} = \mathcal{F}_{4123} - \mathcal{F}_{4132} - \mathcal{F}_{4213} + \mathcal{F}_{4321} \,.$$

1.3. **The descent map $\mathcal{D}\colon \mathfrak{S}Sym \to \mathcal{Q}Sym$.** The descent set $\mathrm{Des}(u)$ of a permutation $u \in \mathfrak{S}_n$ is the subset of $[n{-}1]$ recording the descents of $u$

$$\mathrm{Des}(u) := \{p \in [n{-}1] \mid u_p > u_{p+1}\} \,. \tag{1.10}$$

Thus $\mathrm{Des}(46512837) = \{2, 3, 6\}$. Malvenuto [15, théorèmes 5.12, 5.13, and 5.18] shows that there is a morphism of Hopf algebras

$$\begin{array}{rcccl} \mathcal{D} & : & \mathfrak{S}Sym & \longrightarrow & \mathcal{Q}Sym \\ & & \mathcal{F}_u & \longmapsto & F_{\mathrm{Des}(u)} \end{array} \tag{1.11}$$

1.4. **Galois connections $\mathfrak{S}_n \leftrightarrows \mathcal{Q}_n$.** Underlying our results are combinatorial facts concerning the lattices $\mathfrak{S}_n$ and $\mathcal{Q}_n$. We describe two important conceptual facts. For a subset $\mathsf{S} \subseteq [n{-}1]$, let $\mathfrak{S}_{\mathsf{S}} \subseteq \mathfrak{S}_n$ be the parabolic subgroup

$$\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_k} \,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is the composition of $n$ such that $I(\alpha) = \mathsf{S}$. For a subset $\mathsf{S} \subseteq [n{-}1]$, let $Z(\mathsf{S}) \in \mathfrak{S}_n$ be the maximal permutation with descent set $\mathsf{S}$.

A *Galois connection* between posets $P$ and $Q$ is a pair $(f, g)$ of order preserving maps $f\colon P \to Q$ and $g\colon Q \to P$ such that for any $x \in P$ and $y \in Q$,

$$f(x) \leq y \iff x \leq g(y) \,. \tag{1.12}$$

Equivalently, the map $f$ is left adjoint to the map $g$.

**Proposition 1.1.** *The pair of maps* $(\mathrm{Des}, Z) : \mathfrak{S}_n \rightleftarrows \mathcal{Q}_n$ *is a Galois connection.*

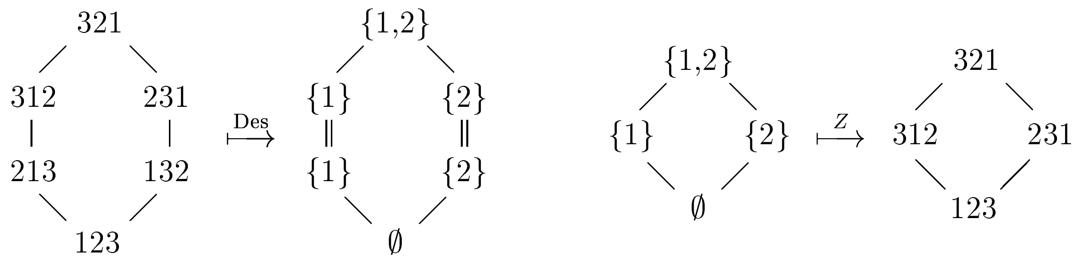


FIGURE 2. The Galois connection $\mathfrak{S}_3 \rightleftarrows \mathcal{Q}_3$

This Galois connection is why the monomial basis of $\mathfrak{S}Sym$ is analogous to that of $\mathcal{Q}Sym$, and is why we consider the weak order on $\mathfrak{S}_n$. The connection between the monomial bases of these two algebras will be elucidated in Theorem 3.4.

A permutation $u \in \mathfrak{S}_n$ has a *global descent* at a position $p \in [n{-}1]$ if

$$i \le p < j \implies u_i > u_j\,.$$

Equivalently, if $\{u_1, \ldots, u_p\} = \{n, n{-}1, \ldots, n{-}p{+}1\}$. Let $\mathrm{GDes}(u) \subseteq [n{-}1]$ be the set of global descents of $u$. Note that $\mathrm{GDes}(u) \subseteq \mathrm{Des}(u)$, but these are not equal in general.

The notion of global descents is a very natural companion of that of (ordinary) descents, in that the map $\mathrm{GDes}\colon \mathfrak{S}_n \to \mathcal{Q}_n$ is *right* adjoint to $Z\colon \mathcal{Q}_n \to \mathfrak{S}_n$.

**Proposition 1.2.** *The pair of maps* $(Z, \mathrm{GDes}) : \mathcal{Q}_n \rightleftarrows \mathfrak{S}_n$ *is a Galois connection.*

## 2. Elementary Algebraic Structure of $\mathfrak{S}Sym$

2.1. **The coproduct of $\mathfrak{S}Sym$.** The coproduct of $\mathfrak{S}Sym$ (1.7) takes a simple form on the monomial basis. For a permutation $u \in \mathfrak{S}_n$, define $\overline{\mathrm{GDes}}(u)$ to be $\mathrm{GDes}(u) \cup \{0, n\}$.

**Theorem 2.1.** *Let* $u \in \mathfrak{S}_n$. *Then*

$$\Delta(\mathcal{M}_u) \;=\; \sum_{p \in \overline{\mathrm{GDes}}(u)} \mathcal{M}_{\mathrm{st}(u_1,\ldots,u_p)} \otimes \mathcal{M}_{\mathrm{st}(u_{p+1},\ldots,u_n)}\,. \tag{2.1}$$

2.2. **The product of $\mathfrak{S}Sym$.** The product of $\mathfrak{S}Sym$ in terms of its monomial basis has non-negative structure constants, which we describe. For instance,

$$\begin{aligned} \mathcal{M}_{12} \cdot \mathcal{M}_{21} \;=\; & \mathcal{M}_{4312} + \mathcal{M}_{4231} + \mathcal{M}_{3421} + \mathcal{M}_{4123} + \mathcal{M}_{2341} \\ & + \mathcal{M}_{1243} + \mathcal{M}_{1423} + \mathcal{M}_{1342} + 3\mathcal{M}_{1432} + 2\mathcal{M}_{2431} + 2\mathcal{M}_{4132}\,. \end{aligned} \tag{2.2}$$

First, for a Grassmannian permutation $\zeta \in \mathfrak{S}^{(p,q)}$ (a left coset representative of $\mathfrak{S}_p \times \mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$), consider the map corresponding to the *right coset* of $\zeta^{-1}$.

$$\rho_\zeta \;:\; \mathfrak{S}_p \times \mathfrak{S}_q \;\to\; \mathfrak{S}_{p+q}\,, \qquad \rho_\zeta(u,v) \;:=\; (u \times v)\cdot \zeta^{-1}\,.$$

This order-preserving map is injective and its image is an interval in $\mathfrak{S}_{p+q}$. For $u \in \mathfrak{S}_p$, $v \in \mathfrak{S}_q$ and $w \in \mathfrak{S}_{p+q}$, define $A^w_{u,v} \subseteq \mathfrak{S}^{(p,q)}$ to be

$$A^w_{u,v} \;=\; \left\{\zeta \in \mathfrak{S}^{(p,q)} \mid (u,v) = \max \rho_\zeta^{-1}[1,w]\right\}, \tag{2.3}$$

where $[w, w'] := \{w'' \mid w \leq w'' \leq w'\}$ denotes the interval between $w$ and $w'$. This set has another description as the set of those $\zeta \in \mathfrak{S}^{(p,q)}$ satisfying

$$\text{(2.4)} \qquad \begin{array}{ll} (i) & (u \times v) \cdot \zeta^{-1} \leq w, \text{ and} \\ (ii) & \text{if } u \leq u' \text{ and } v \leq v' \text{ satisfy } (u' \times v') \cdot \zeta^{-1} \leq w, \\ & \text{then } u = u' \text{ and } v = v'. \end{array}$$

Set $\alpha^w_{u,v} := \# A^w_{u,v}$.

**Theorem 2.2.** *For any $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, we have*

$$\mathcal{M}_u \cdot \mathcal{M}_v \;=\; \sum_{w \in \mathfrak{S}_{p+q}} \alpha^w_{u,v}\, \mathcal{M}_w\,. \tag{2.5}$$

For instance, in (2.2) the coefficient of $\mathcal{M}_{2431}$ in $\mathcal{M}_{12} \cdot \mathcal{M}_{21}$ is 2 because among the six permutations in $\mathfrak{S}^{(2,2)}$,

$$1234,\ 1324,\ 1423,\ 2314,\ 2413,\ 3412\,,$$

only the first two satisfy conditions $(i)$ and $(ii)$ of (2.4).

The structure constants $\alpha^w_{u,v}$ admit a geometric-combinatorial description in terms of the permutahedron. The vertices of the $(n-1)$-dimensional permutahedron are indexed by the elements of $\mathfrak{S}_n$ so that the 1-skeleton is the Hasse diagram of the weak order (see Figure 1). Facets of the permutahedron are products of two lower dimensional permutahedra, and the image of $\rho_\zeta$ is the set of vertices in a facet. Moreover, every facet arises in this way for a unique triple $(p, q, \zeta)$ with $p + q = n$ and $\zeta \in \mathfrak{S}^{(p,q)}$ (see [5, Exer. 2.9]). Such such a facet has *type* $(p, q)$. Figure 3 shows the image of $\rho_{1324}$, a facet of the 3-permutahedron of type $(2, 2)$, and the permutation 2431.

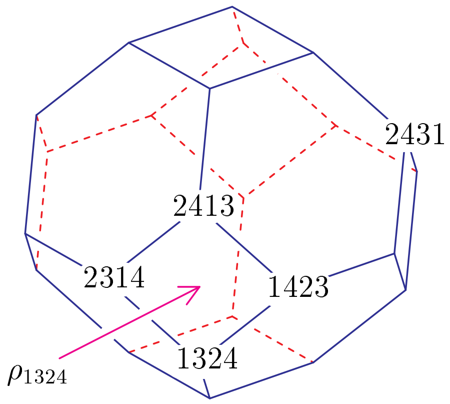


Figure 3. The facet $\rho_{1324}$ of type $(2, 2)$ and $w = 2431$.

The description (2.3) of $A^w_{u,v}$ (and hence of $\alpha^w_{u,v}$) can be interpreted as follows: Given $u \in \mathfrak{S}_p$, $v \in \mathfrak{S}_q$, and $w \in \mathfrak{S}_{p+q}$, the structure constant $\alpha^w_{u,v}$ counts the number of facets of type $(p, q)$ of the $(p+q-1)$-permutahedron such that the vertex $\rho_\zeta(u, v)$ is below $w$ and it is the maximum vertex in that facet below $w$.

For instance, the facet $\rho_{1324}$ contributes to the structure constant $\alpha^{2431}_{12,21}$ because the vertex $\rho_{1324}(12, 21) = 1423$ satisfies the required properties in relation to the vertex $w = 2431$, as shown in Figure 3.

2.3. **The antipode of $\mathfrak{S}Sym$.** Malvenuto left open the problem of an explicit formula for the antipode of $\mathfrak{S}Sym$ [15, pp. 59-60]. We identify the coefficients of the antipode in terms of both bases in explicit combinatorial terms. These are based upon a general formula for the antipode of a connected Hopf algebra due to Milnor and Moore [17].

For any subset $\mathsf{S} = \{p_1 < p_2 < \cdots < p_k\} \subseteq [n{-}1]$ and $v \in \mathfrak{S}_n$ set

$$v_{\mathsf{S}} \;:=\; \mathrm{st}(v_1,\ldots,v_{p_1}) \times \mathrm{st}(v_{p_1+1},\ldots,v_{p_2}) \times \cdots \times \mathrm{st}(v_{p_k+1},\ldots,v_n) \;\in\; \mathfrak{S}_{\mathsf{S}}\,.$$

**Theorem 2.3.** *For $v, w \in \mathfrak{S}_n$ set*

$$\begin{aligned}\lambda(v,w) \;:=\;& \#\{\mathsf{S} \subseteq [n{-}1] \mid \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is odd}\}\\ &- \#\{\mathsf{S} \subseteq [n{-}1] \mid \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is even}\}.\end{aligned}$$

*Then*

$$S(\mathcal{F}_v) \;=\; \sum_{w\in\mathfrak{S}_n} \lambda(v,w)\,\mathcal{F}_w\,. \tag{2.6}$$

These coefficients of the antipode may indeed be positive or negative. For instance,

$$S(\mathcal{F}_{231}) \;=\; \mathcal{F}_{132} - \mathcal{F}_{213} - 2\mathcal{F}_{231} + \mathcal{F}_{312}\,.$$

The coefficient of $\mathcal{F}_{312}$ is 1 because $\{1\}$, $\{2\}$, and $\{1,2\}$ are the subsets $\mathsf{S}$ of $\{1,2\}$ which satisfy $\mathrm{Des}\big((312)^{-1}(231)_{\mathsf{S}}\big) \subseteq \mathsf{S}$.

Our description of these coefficients is semi-combinatorial, in the sense that it involves a difference of cardinalities of sets. On the monomial basis the situation is different. The sign of the coefficients of $S(\mathcal{M}_v)$ only depends on the number of global descents of $v$. We provide a fully combinatorial description of these coefficients. Let $v, w \in \mathfrak{S}_n$ and suppose $\mathsf{S} \subseteq \mathrm{GDes}(v)$. Define $C_{\mathsf{S}}(v,w) \subseteq \mathfrak{S}^{\mathsf{S}}$ to be those $\zeta \in \mathfrak{S}^{\mathsf{S}}$ satisfying

$$\begin{array}{rl} (i) & v_{\mathsf{S}}\zeta^{-1} \le w,\\ (ii) & \text{if } v \le v' \text{ and } v'_{\mathsf{S}}\zeta^{-1} \le w \text{ then } v = v', \text{ and}\\ (iii) & \text{if } \mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq \mathsf{S} \text{ and } v_{\mathsf{R}}\zeta^{-1} \le w \text{ then } \mathsf{R} = \mathsf{S}. \end{array} \tag{2.7}$$

Set $\kappa(v,w) := \#C_{\mathrm{GDes}(v)}(v,w)$.

**Theorem 2.4.** *For $v, w \in \mathfrak{S}_n$, we have*

$$S(\mathcal{M}_v) \;=\; (-1)^{\#\mathrm{GDes}(v)+1} \sum_{w\in\mathfrak{S}_n} \kappa(v,w)\,\mathcal{M}_w\,. \tag{2.8}$$

For instance,

$$\begin{aligned} S(\mathcal{M}_{3412}) \;=\; & \mathcal{M}_{1234} + 2\mathcal{M}_{1324} + \mathcal{M}_{1342} + \mathcal{M}_{1423}\\ &+ \mathcal{M}_{2314} + \mathcal{M}_{2413} + \mathcal{M}_{3124} + \mathcal{M}_{3142} + \mathcal{M}_{3412}\,. \end{aligned}$$

Consider the coefficient of $\mathcal{M}_{3412}$. In this case, $\mathsf{S} = \mathrm{GDes}(3412) = \{2\}$, so

$$\mathfrak{S}^{\mathsf{S}} \;=\; \{1234,\ 1324,\ 1423,\ 2314,\ 2413,\ 3412\}\,.$$

We invite the reader to verify that 3412 is the only element of $\mathfrak{S}^{\{2\}}$ that satisfies all three conditions of (2.7). Therefore $C_{\mathsf{S}}(3412, 3412) = \{3412\}$ and the coefficient is $\kappa(3412, 3412) = 1$.

**Remark 2.5.** The antipode of $\mathfrak{S}Sym$ has infinite order. A computation gives that

$$S^{2m}(\mathcal{M}_{231}) \;=\; \mathcal{M}_{231} + 2m(\mathcal{M}_{213} - \mathcal{M}_{132}) \qquad \forall\, m \in \mathbb{Z}\,.$$

## 3. Hopf-algebraic structure of $\mathfrak{S}Sym$

3.1. **Cofreeness of $\mathfrak{S}Sym$ and the coradical filtration.** The basis $\{\mathcal{M}_u\}$ reveals the existence of a second coalgebra grading on $\mathfrak{S}Sym$, given by the number of global descents. With respect to this grading, $\mathfrak{S}Sym$ is a cofree graded coalgebra. We deduce an elegant description of the coradical filtration: it corresponds to a filtration of the symmetric groups by certain lower order ideals determined by the number of global descents. In particular, the space of primitive elements is spanned by those $\mathcal{M}_u$ where $u$ has no global descents.

Let $V$ be a vector space and consider the graded vector space

$$Q(V) \;:=\; \bigoplus_{k\geq 0} V^{\otimes k}\,.$$

The space $Q(V)$ is a graded connected coalgebra under the *deconcatenation* coproduct

$$\Delta(v_1 \otimes \cdots \otimes v_k) \;=\; \sum_{i=0}^{k} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k)\,,$$

and counit $\epsilon(v_1 \otimes \cdots \otimes v_k) = 0$ if $k \geq 1$.

The following universal property is satisfied for the canonical projection $\pi : Q(V) \to V$. Given a graded coalgebra $C = \oplus_{k\geq 0} C^k$ and a linear map $\varphi : C \to V$ such that $\varphi(C^k) = 0$ for $k \neq 1$, there is a unique morphism of graded coalgebras $\hat{\varphi} : C \to Q(V)$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, dashed, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & & Q(V) \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Explicitly, $\hat{\varphi}$ is defined by

$$\hat{\varphi}|_{C^k} = \varphi^{\otimes k} \Delta^{(k-1)}\,. \tag{3.1}$$

In particular, $\hat{\varphi}|_{C^0} = \epsilon$, $\hat{\varphi}|_{C^1} = \varphi$ and $\hat{\varphi}|_{C^2} = (\varphi \otimes \varphi)\Delta$.

To establish the cofreeness of $\mathfrak{S}Sym$, we first define a second coalgebra grading. Let $\mathfrak{S}^0 := \mathfrak{S}_0$, and for $k \geq 1$, let

$$\begin{aligned} \mathfrak{S}^k_n \;&:=\; \{u \in \mathfrak{S}_n \mid u \text{ has exactly } k{-}1 \text{ global descents}\}, \quad \text{and} \\ \mathfrak{S}^k \;&:=\; \coprod_{n\geq 0} \mathfrak{S}^k_n\,. \end{aligned}$$

For instance,

$$\begin{aligned} \mathfrak{S}^1 \;=\; \{1\} \cup \{12\} \cup \{123,\, 213,\, 132\} \cup \{&1234,\, 2134,\, 1324,\, 1243,\, 3124, \\ &2314,\, 2143,\, 1423,\, 1342,\, 3214,\, 3142,\, 2413,\, 1432\} \cup \cdots \end{aligned}$$

Let $(\mathfrak{S}Sym)^k$ be the vector subspace of $\mathfrak{S}Sym$ spanned by $\{\mathcal{M}_u \mid u \in \mathfrak{S}^k\}$.

**Theorem 3.1.** *The decomposition $\mathfrak{S}Sym = \oplus_{k\geq 0}(\mathfrak{S}Sym)^k$ is a coalgebra grading. Moreover, endowed with this grading, $\mathfrak{S}Sym$ is a cofree graded coalgebra.*

The coradical $C^{(0)}$ of a graded connected coalgebra $C$ is the 1-dimensional component in degree 0 (identified with the base field via the counit). The primitive elements are

$$\mathrm{Prim}(C) \;:=\; \{x \in C \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}\,.$$

Set $C^{(1)} := C^{(0)} \oplus \mathrm{Prim}(C)$, the first level of the coradical filtration. More generally, the $k$-th level of the coradical filtration is

$$C^{(k)} \;:=\; \bigl(\Delta^{(k)}\bigr)^{-1}\Bigl(\sum_{i+j=k} C^{\otimes i} \otimes C^{(0)} \otimes C^{\otimes j}\Bigr)\,.$$

We have $C^{(0)} \subseteq C^{(1)} \subseteq C^{(2)} \subseteq \cdots \subseteq C = \bigcup_{k\geq 0} C^{(k)}$, and

$$\Delta(C^{(k)}) \;\subseteq\; \sum_{i+j=k} C^{(i)} \otimes C^{(j)}\,.$$

Thus, the coradical filtration measures the complexity of iterated coproducts.

For a cofree graded coalgebra $Q(V)$, the coradical filtration is easy to describe. The space of primitive elements is just $V$, and the $k$-th level of the coradical filtration is $\oplus_{i=0}^{k} V^{\otimes i}$. These are immediate from the definition of the deconcatenation coproduct.

Define

$$\mathfrak{S}_n^{(k)} \;:=\; \coprod_{i=0}^{k} \mathfrak{S}_n^{k} \quad\text{and}\quad \mathfrak{S}^{(k)} \;:=\; \coprod_{i=0}^{k} \mathfrak{S}^{k}\,.$$

In other words, $\mathfrak{S}^{(0)} = \mathfrak{S}_0$ and for $k \geq 1$,

$$\mathfrak{S}_n^{(k)} = \{u \in \mathfrak{S}_n \mid u \text{ has at most } k{-}1 \text{ global descents}\}\,.$$

Proposition 1.2 asserts that GDes: $\mathfrak{S}_n \to \mathcal{Q}_n$ is order-preserving. Since $\mathcal{Q}_n$ is ranked by the cardinality of a subset, it follows that $\mathfrak{S}_n^{(k)}$ is a lower order ideal of $\mathfrak{S}_n$, with $\mathfrak{S}_n^{(k)} \subseteq \mathfrak{S}_n^{(k+1)}$. The coradical filtration corresponds precisely to this filtration of the symmetric groups by lower ideals.

**Corollary 3.2.** *A linear basis for the $k$-th level of the coradical filtration of $\mathfrak{S}Sym$ is*

$$\{\mathcal{M}_u \mid u \in \mathfrak{S}^{(k)}\}\,.$$

*In particular, a linear basis for the space of primitive elements is*

$$\{\mathcal{M}_u \mid u \textit{ has no global descents}\}\,.$$

3.2. **The descent map to quasi-symmetric functions.** We study the effect of the morphism of Hopf algebras (1.11)

$$\mathcal{D} \,:\, \mathfrak{S}Sym \;\to\; \mathcal{Q}Sym, \qquad \text{defined by} \qquad \mathcal{F}_u \;\mapsto\; F_{\mathrm{Des}(u)}$$

on the monomial basis. Here, we use subsets $\mathsf{S}$ of $[n{-}1]$ to index monomial quasi-symmetric functions of degree $n$.

**Definition 3.3.** A permutation $u \in \mathfrak{S}_n$ is *closed* if we have $u = Z(\mathsf{T})$ for some $\mathsf{T} \in \mathcal{Q}_n$. Equivalently, $u$ is closed if and only if $\mathrm{Des}(u) = \mathrm{GDes}(u)$.

**Theorem 3.4.** *Let $u \in \mathfrak{S}_n$. Then*

$$\mathcal{D}(\mathcal{M}_u) \;=\; \begin{cases} M_{\mathrm{GDes}(u)} & \textit{if } u \textit{ is closed,}\\ 0 & \textit{if not.}\end{cases}$$

3.3. **$\mathfrak{S}Sym$ as a crossed product over $\mathcal{Q}Sym$.** We describe the *algebra* structure of $\mathfrak{S}Sym$ as a crossed product over the Hopf algebra $\mathcal{Q}Sym$. See [18, §7] for a review of this construction in the general Hopf algebraic setting. Let us only say that the crossed product of a Hopf algebra $K$ with an algebra $A$ with respect to a Hopf cocycle $\sigma : K{\otimes}K \to A$ is a certain algebra structure on the space $A \otimes K$, denoted by $A\#_\sigma K$.

**Proposition 3.5.** *The map $M_{\mathsf{S}} \mapsto \mathcal{M}_{Z(\mathsf{S})}$ induces a morphism of coalgebras $\mathcal{Z}\colon \mathcal{Q}Sym \to \mathfrak{S}Sym$ that is a right inverse to the morphism of Hopf algebras $\mathcal{D}\colon \mathfrak{S}Sym \to \mathcal{Q}Sym$.*

In this situation, an important theorem of Blattner, Cohen and Montgomery [6] applies. Namely, suppose $\pi : H \to K$ is a morphism of Hopf algebras that admits a coalgebra splitting $\gamma : K \to H$. Then there is a *crossed product* decomposition

$$H \;\cong\; A\#_\sigma K$$

where $A$ is the *left Hopf kernel* of $\pi$:

$$A \;=\; \{h \in H \mid \sum h_1{\otimes}\pi(h_2) = h{\otimes}1\}$$

and the *Hopf cocycle* $\sigma : K{\otimes}K \to A$ is

$$\sigma(k, k') = \sum \gamma(k_1)\gamma(k'_1)S\gamma(k_2k'_2)\,.$$

Note that if $\pi$ and $\gamma$ preserve gradings, then so does the rest of the structure.

Let $A$ be the left Hopf kernel of $\mathcal{D}\colon \mathfrak{S}Sym \to \mathcal{Q}Sym$ and $A_n$ its $n$-th homogeneous component. Once again the monomial basis of $\mathfrak{S}Sym$ proves useful in describing $A$.

**Proposition 3.6.** *A basis for $A_n$ is the set $\{\mathcal{M}_u\}$ where $u$ runs over all permutations of $n$ that are not of the form*

$$* \dots * \, 12 \dots n{-}k \tag{3.2}$$

*for any $k = 0, \dots, n-1$. In particular,*

$$\dim A_n \;=\; n! - \sum_{k=0}^{n-1} k!\,.$$

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