APPENDICES TO EXPLICIT ENUMERATIVE GEOMETRY FOR THE REAL GRASSMANNIAN OF LINES IN PROJECTIVE SPACE

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ABSTRACT. We illustrate our results in the first case which is not covered by the exact solutions given in §8.

9. APPENDIX: THE 5 LINES MEETING 6 $\mathbf{P}^2$'S IN $\mathbf{P}^4$.

By Corollary 3.3, the number of lines meeting 6 generic $\mathbf{P}^2$'s in $\mathbf{P}^4$ equals the number of tableaux of shape $(3,3)$. This number is 5 as

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}$$

are all tableaux of shape $(3,3)$. By Theorem B, there are six real $\mathbf{P}^2$'s with five real lines incident on all six. Here, we use our methods to construct a family of sextuples of $\mathbf{P}^2$'s in $\mathbf{P}^4$ such that for small real values of some parameters, we obtain six real $\mathbf{P}^2$'s with five real lines incident on all six.

9.1. **Arrangements.** We wish to construct arrangements as simply as possible over $\mathbf{R}$. To that end, we use the ideas of §7.3 to find arrangements whose hyperplanes are given by linear forms with small integral coefficients. Arrangements may be constructed inductively; to find an arrangement in $\mathbf{P}^n$, it suffices to begin with one in $\mathbf{P}^{n-1}$, and then select the last two hyperplanes. We illustrate this by inductively finding an arrangement in $\mathbf{P}^5$.

Let $x_1,\ldots,x_6$ be coordinates for $\mathbf{P}^5$ and $F_{\bullet}$ be the flag with $F_i : x_1 = \cdots = x_i = 0$.

$\mathbf{P}^2$: We first select H_2, a hyperplane containing F_2. Thus H_2 has equation $Ax_1 + Bx_2$ for some A, B. Let $A = 0$ and $B = 1$, a choice with small integral coefficients, gives $H_2 : x_2 = 0$.

$\mathbf{P}^3$: H_3 contains F_2 and is distinct from H_2. It suffices to take $H_3 : x_1 = 0$. Since $H_4 \bigcap F_2 = F_3$, it suffices to take $H_4 : x_3 = 0$.

$\mathbf{P}^4$: H_5 must contain F_3, but none of the three codimension 2 planes:

$$H_{2\,3} : x_1 = x_2 = 0 \qquad H_{2\,4} : x_2 = x_3 = 0 \qquad H_{3\,4} : x_1 = x_3 = 0.$$

Thus H_5 has an equation $Ax_1 + Bx_2 + Cx_3$ with $A \cdot B \cdot C \neq 0$. Among all possible choices, $H_5 : x_1 + x_2 + x_3 = 0$ has the smallest integral coefficients.

Since $H_6 \bigcap F_3 = F_4$, it suffices to take $H_6 : x_4 = 0$.

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$\mathbf{P}^5$: H_7 must contain F_4, but none of the seven 2-planes H_A for

$$A \in \{234;\ 236;\ 246,\ 256,\ 346,\ 356,\ 456\}.$$

Letting $[a\ b\ c]$ be coordinates for $\mathbf{P}^2$, these 7 $\mathbf{P}^2$'s have coordinates

$$\begin{array}{rcl}
H_{234} &=& [\begin{array}{cccccc}\cdot & \cdot & \cdot & a & b & c\end{array}]\\
H_{236} &=& [\begin{array}{cccccc}\cdot & \cdot & a & \cdot & b & c\end{array}]\\
H_{246} &=& [\begin{array}{cccccc}a & \cdot & \cdot & \cdot & b & c\end{array}]\\
H_{256} &=& [\begin{array}{cccccc}a & \cdot & -a & \cdot & b & c\end{array}]\\
H_{346} &=& [\begin{array}{cccccc}\cdot & a & \cdot & \cdot & b & c\end{array}]\\
H_{356} &=& [\begin{array}{cccccc}\cdot & a & -a & \cdot & b & c\end{array}]\\
H_{456} &=& [\begin{array}{cccccc}a & -a & \cdot & \cdot & b & c\end{array}],
\end{array}$$

where $\cdot$ has been used in place of 0.

Since H_7 must contain F_4, it has equation $Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0$. The conditions on H_7 amount to the conditions that $A\cdot B\cdot C\cdot D \neq 0$ and A, B, C are distinct. We can solve this system in $\mathbf{Z}/5\mathbf{Z}$: If we normalize so that $D = 1$, then there are 24 choices for A, B, C, and 6 if we want A, B, C (as integers) to be as small as possible. We select $H_7 : x_1 + 2x_2 + 3x_3 + x_4 = 0$.

Since $H_8 \bigcap F_4 = F_5$, it suffices to take $H_8 : x_5 = 0$.

Question: Find a nice way to iterate this procedure; or an algorithm (involving tableaux?) to inductively construct arrangements in $\mathbf{P}^n$.

9.2. **The 5 lines meeting 6 $\mathbf{P}^2$'s in $\mathbf{P}^4$.** We list the hyperplanes in the arrangement constructed in §1.1 for $\mathbf{P}^4$:

$$\begin{array}{rcll}
H_2 &=& [\begin{array}{ccccc}* & \cdot & * & * & *\end{array}] & \qquad x_2 = 0\\
H_3 &=& [\begin{array}{ccccc}\cdot & * & * & * & *\end{array}] & \qquad x_1 = 0\\
H_4 &=& [\begin{array}{ccccc}* & * & \cdot & * & *\end{array}] & \qquad x_3 = 0\\
H_5 &=& [\begin{array}{ccccc}-a & -b & a+b & * & *\end{array}] & \qquad x_1 + x_2 + x_3 = 0\\
H_6 &=& [\begin{array}{ccccc}* & * & * & \cdot & *\end{array}] & \qquad x_4 = 0,
\end{array}$$

where $*, a, b$ are arbitrary numbers and $\cdot$ is 0.

In Theorem C', we showed that $\Omega(\mathcal{T}_6) = \{H_{246}, H_{256}, H_{346}, H_{356}, H_{456}\}$, a degree 5 reduced zero cycle, is in the closure of the set of degree 5 zero cycles on $\mathbf{G}_1\mathbf{P}^4$ which meet six real $\mathbf{P}^2$'s in general position, considered as a subset of $S^5\mathbf{G}_1\mathbf{P}^4(\mathbf{R})$.

We construct a 7-parameter family of sextuples of real $\mathbf{P}^2$'s such that $\Omega(\mathcal{T}_6)$ is in the closure of the set of degree 5 zero cycles on $\mathbf{G}_1\mathbf{P}^4$ which meet sextuples in this family.

Let K_1—K_6 be $\mathbf{P}^2$'s depending upon parameters $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ with coordinates and defining ideals:

$$\begin{array}{rcllllll l}
K_1 &=& [\ \cdot & \cdot & a & b & c\ ] & & (x_1, x_2) \\
K_2 &=& [\ a & \alpha b & \cdot & b & c\ ] & & (x_3, x_2 - \alpha x_4) \\
K_3 &=& [-\beta a & b & \gamma c & \gamma a - c & -a\ ] & & (x_1 - \beta x_5, x_3 + \gamma x_4 + \gamma^2 x_5) \\
K_4 &=& [\ a & b & \delta c & \lambda c & -c\ ] & & (x_3 - \delta x_5, x_4 + \lambda x_5) \\
K_5 &=& [-a & b & a - b & \mu c - b & -c\ ] & & (x_1 + x_2 + x_3, x_2 + x_4 + \mu x_5) \\
K_6 &=& [\ a & b & -c & \nu c & \cdot\ ] & & (\nu x_3 + x_4, x_5).
\end{array}$$

Let Ω_{K_i} be the Schubert variety of lines meeting K_i. We claim:

1. For general values of the parameters $\alpha, \beta, \ldots, \nu$, the intersection
$$\Omega_{K_1} \bigcap \Omega_{K_2} \bigcap \Omega_{K_3} \bigcap \Omega_{K_4} \bigcap \Omega_{K_5} \bigcap \Omega_{K_6}$$
is a generically transverse intersection.
2. For sufficiently small values of α, β, γ and δ, and all values of λ, μ, ν, this intersection consists of 5 real lines in $\mathbf{P}^4$.

9.3. **Defining Ideals.** The Grassmannian $\mathbf{G}_1\mathbf{P}^4$ is embedded into $\mathbf{P}(\Lambda^2\mathbf{C}^5) = \mathbf{P}^9$ via the Plücker embedding. The coordinates x_1, x_2, x_3, x_4, x_5 of $\mathbf{P}^4$ give rise to Plücker coordinates $\{[i\,j] \mid 1 \leq i < j \leq 5\}$ for $\mathbf{P}^9$, where $[i\,j] \leftrightarrow x_i \wedge x_j$. The ideal of $\mathbf{G}_1\mathbf{P}^4$ in $\mathbf{P}^9$ is given by the Plücker equations

$$\begin{gathered}
[12]\,[34] - [13]\,[24] + [14]\,[23] \\
[12]\,[35] - [13]\,[25] + [15]\,[23] \\
[12]\,[45] - [14]\,[25] + [15]\,[24] \\
[13]\,[45] - [14]\,[35] + [15]\,[34] \\
[23]\,[45] - [24]\,[35] + [25]\,[34]
\end{gathered}$$

Let K be $\mathbf{P}^2$ defined by two linear forms A and B. Then the set of lines meeting K in $\mathbf{P}^4$ is a hyperplane section of $\mathbf{G}_1\mathbf{P}^4$ in $\mathbf{P}^9$, the hyperplane defined by the form $A \wedge B$. Thus K_1—K_6 induce the following hyperplanes in $\mathbf{P}^9$:

$$\begin{array}{rcl}
K_1 &:& [12] \\
K_2 &:& [23] + \alpha[34] \\
K_3 &:& [13] + \beta[35] + \beta\gamma[45] + \gamma[14] + \gamma^2[15] \\
K_4 &:& [34] + \delta[45] + \lambda[35] \\
K_5 &:& [12] + [14] - [23] + [24] + [34] + \mu([15] + [25] + [35]) \\
K_6 &:& [45] + \nu[35]
\end{array}$$

Let $\mathcal{I}(\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu)$ be the ideal in $\mathbf{P}^9$ generated by the Plücker equations and these six linear forms. In §8.4 we use geometry to show that for a generic choice of the parameters $\alpha, \ldots, \nu$, the ideal $\mathcal{I}(\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu)$ defines a reduced zero-dimensional scheme of degree 5. For these parameters, the intersection

$$\Omega_{K_1} \bigcap \Omega_{K_2} \bigcap \Omega_{K_3} \bigcap \Omega_{K_4} \bigcap \Omega_{K_5} \bigcap \Omega_{K_6}$$

is generically transverse, proving assertion 1 from the end of §8.2. These methods will also prove assertion 2, that for certain real values of the parameters, all points of intersection are real.

9.4. **Geometry.** If $F \subsetneq H$ is a partial flag in $\mathbf{P}^4$, let $\Omega(F, H)$ be those lines in H which also meet F. If $H = \mathbf{P}^4$, write Ω_F for this Schubert variety and, if F has codimension 1 in H, then $\Omega(F, H)$ is all the lines in H, $\mathbf{G}_1 H$.

If T is a tableau whose first row has length $a \leq 3$, then $\Omega T = \Omega(F_{a+1}, H_T)$, where F_{a+1} is in the flag $F_\bullet$ and H_T is the intersection of those H_j where j is in the second row of T.

Let $\mathcal{T}_i$ be the set of all Young tableaux of degree i whose first row has length at most 3. For example, $\mathcal{T}_4$ is

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$$

We now study the cycles $\Omega_{K_1} \bigcap \Omega_{K_2} \bigcap \Omega_{K_3} \bigcap \Omega_{K_4} \bigcap \Omega_{K_5} \bigcap \Omega_{K_6}$ as the parameters $\alpha, \ldots, \nu$ vary. Our aim is to use geometric arguments to verify the assertions 1 and 2 from the end of §8.2.

$\boldsymbol{\alpha}$: If $\alpha \neq 0$, then K_1 and $K_2(\alpha)$ meet properly, so $\Omega_{K_1} \bigcap \Omega_{K_2(\alpha)}$ is generically transverse and irreducible.

When $\alpha = 0$, then $K_1 \bigcap K_2(\alpha) = F_3$ and $\langle K_1, K_2(\alpha) \rangle = H_2$. This and other such statements in this section are easily verified by examining the defining equations of K_1–K_6 from §8.2. Thus, by Lemma 2.4,

$$\Omega_{K_1} \bigcap \Omega_{K_2(\alpha)} \;=\; \Omega_{F_3} + \mathbf{G}_1 H_2 \;=\; \Omega_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} + \Omega_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} \;=\; \Omega(\mathcal{T}_2).$$

Thus

1. If $\beta, \ldots, \nu$ are chosen so that $\Omega(\mathcal{T}_2)$, $\Omega_{K_3}, \ldots, \Omega_{K_6}$ meet generically transversally, then for general α, the Schubert varieties $\Omega_{K_1}, \Omega_{K_2}, \ldots, \Omega_{K_6}$ meet generically transversally.
2. If $\beta, \ldots, \nu$ are chosen so that $\Omega(\mathcal{T}_2) \bigcap \Omega_{K_3} \bigcap \cdots \bigcap \Omega_{K_6}$ is 5 real lines, then for α sufficiently close to 0, $\Omega_{K_1} \bigcap \Omega_{K_2} \bigcap \cdots \bigcap \Omega_{K_6}$ is 5 real lines.

$\boldsymbol{\beta}$: For any (β, γ), $K_3(\beta, \gamma)$ meets H_2 properly, and if $\beta \cdot \gamma \neq 0$, then $K_3 \bigcap F_3 = \emptyset$. Thus if $\beta \cdot \gamma \neq 0$, Ω_{K_3} meets $\Omega(\mathcal{T}_2)$ generically transversally.

If $\beta = 0$ and $\gamma \neq 0$, then $K_3(0, \gamma)$ meets F_3 in a point, $F_4(\gamma)$, and $\langle K_3, F_3 \rangle = H_3$. Let $F_3(\gamma) = K_3 \bigcap H_3$. Then

$$\begin{aligned}\Omega(\mathcal{T}_2) \bigcap \Omega_{K_3(\beta,\gamma)} \;&=\; \Omega_{K_3} \bigcap \mathbf{G}_1 H_2 + \Omega_{K_3} \bigcap \Omega_{F_3} \\ &=\; \Omega(F_3(\gamma),\, H_2) + \Omega_{F_4(\gamma)} + \Omega(F_3,\, H_3).\end{aligned}$$

Thus

1. If $\gamma,\ldots,\nu$ are chosen so that $\Omega(\mathcal{T}_2)$, $\Omega_{K_3(0,\gamma)}$, $\ldots$, Ω_{K_6} meet generically transversally, then for general β, then the cycles $\Omega(\mathcal{T}_2)$, $\Omega_{K_3(\beta,\gamma)}$, $\ldots$, Ω_{K_6} meet generically transversally.
2. If $\gamma,\ldots,\nu$ are chosen so that $\Omega(\mathcal{T}_2)\bigcap\Omega_{K_3(0,\gamma)}\bigcap\cdots\bigcap\Omega_{K_6}$ is 5 real lines, then for β sufficiently close to 0, then $\Omega(\mathcal{T}_2)\bigcap\Omega_{K_3(\beta,\gamma)}\bigcap\cdots\bigcap\Omega_{K_6}$ is 5 real lines.

$\boldsymbol{\gamma}$**:** Since $F_3\subset K_3(0,0)$, $\Omega(\mathcal{T}_2)\bigcap\Omega_{K_3(0,0)}$ is an improper intersection. However, when $\gamma\neq 0$, let $X(\gamma)=\Omega(\mathcal{T}_2)\bigcap\Omega_{K_3(0,\gamma)}$, which is $\Omega(F_3(\gamma),\,H_2)+\Omega_{F_4(\gamma)}+\Omega(F_3,\,H_3)$. Coordinates and defining ideals for $F_3(\gamma)$ and $F_4(\gamma)$ are:

$$\begin{array}{rcl}F_3(\gamma) &=& [\;\cdot\quad\cdot\quad\gamma a\quad\gamma b-a\quad -b\;] \qquad (x_1,x_2,x_3+\gamma x_4+\gamma^2x_5)\\ F_4(\gamma) &=& [\;\cdot\quad\cdot\quad\cdot\quad\gamma b\quad -b\;] \qquad (x_1,x_2,x_3,x_4+\gamma x_5)\end{array}$$

Thus $F_3(0)=F_3$ and $F_4(0)=F_4$, and so the family $X(\gamma)$ over $\mathbf{R}-\{0\}$ extends to a family over $\mathbf{R}$, whose fibre over 0 is

$$\begin{array}{rcl}X(0) &=& \Omega(F_3,H_2)+\Omega_{F_4}+\Omega(F_3,H_3)\\ &=& \Omega_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}+\Omega_{\begin{smallmatrix}1&2&3\end{smallmatrix}}+\Omega_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} \;=\; \Omega(\mathcal{T}_3).\end{array}$$

Thus

1. If $\delta,\ldots,\nu$ are chosen so that $\Omega(\mathcal{T}_3)$, Ω_{K_4}, Ω_{K_5}, and Ω_{K_6} meet generically transversally, then for general $\gamma\neq 0$, the cycles $\Omega(\mathcal{T}_2)$, $\Omega_{K_3(0,\gamma)}$, $\ldots$, Ω_{K_6} meet generically transversally.
2. If $\gamma,\ldots,\nu$ are chosen so that $\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4}\bigcap\Omega_{K_5}\bigcap\Omega_{K_6}$ is 5 real lines, then for $\gamma\neq 0$ sufficiently close to 0, then $\Omega(\mathcal{T}_2)\bigcap\Omega_{K_3(0,\gamma)}\bigcap\cdots\bigcap\Omega_{K_6}$ is 5 real lines.

$\boldsymbol{\delta}$ **:** For any δ, $K_4(\delta)$ meets both HJ_2 and H_3 properly, and for $\delta\neq 0$, $K_4(\delta)\cap F_3=\emptyset$. Thus if $\delta\neq 0$, $\Omega(\mathcal{T}_3)$ meets $\Omega_{K_4(\delta,\lambda)}$ generically transversally.

If $\delta=0$, then $K_4(0,\lambda)\subset H_4$ and $K_4(0,\lambda)\cap F_3=F_4(\lambda)$. Thus by Lemma 2.4, $\Omega(\mathcal{T}_3)$ meets $\Omega_{K_4(0,\lambda)}$ generically transversally and

$$\begin{array}{rcl}\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4(0,\lambda)} &=& (\Omega(F_3,H_2)+\Omega_{F_4}+\Omega(F_3,H_3))\bigcap\Omega_{K_4(0,\lambda)}\\ &=& \Omega(F_3,H_{24})+\Omega(F_4(\lambda),H_2)+\Omega(F_4,H_4)+\Omega(F_3,H_{34})+\Omega(F_4(\lambda),H_3)\\ &=& \Omega_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}}+\Omega(F_4(\lambda),H_2)+\Omega_{\begin{smallmatrix}1&2&3\\4&&\end{smallmatrix}}+\Omega_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}+\Omega(F_4(\lambda),H_3).\end{array}$$

Thus

1. If λ,μ, and ν are chosen so that $\Omega(\mathcal{T}_3)$, $\Omega_{K_4(0,\lambda)}$, Ω_{K_5}, and Ω_{K_6} meet generically transversally, then for general δ, the cycles $\Omega(\mathcal{T}_3)$, $\Omega_{K_4(\delta,\lambda)}$, Ω_{K_5}, and Ω_{K_6} meet generically transversally.
2. If λ,μ, and ν are chosen so that $\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4(0,\lambda)}\bigcap\Omega_{K_5}\bigcap\Omega_{K_6}$ is five real lines, then for δ sufficiently close to 0, $\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4(\delta,\lambda)}\bigcap\Omega_{K_5}\bigcap\Omega_{K_6}$ is five real lines.

$\boldsymbol{\lambda,\mu,\nu}$: Let M, L_5, L_6 be any three $\mathbf{P}^2$'s which are in linear general position, so that $M\cap L_5$ is a point $p_5 \neq p_6 = M\cap L_6$. Then $\mathbf{G}_1 M \bigcap \Omega_{L_5} \bigcap \Omega_{L_6} = \{\langle p_5, p_6\rangle\}$, a single line. This is real if M, L_5 and L_6 are all real.

Likewise, if $H \simeq \mathbf{P}^3$ and $q\in H$ is a point so that $q \in H$, L_5 and L_6 are in linear general position, then $\Omega(q,H)\bigcap \Omega_{L_5}\bigcap\Omega_{L_6}$ is the single line $\langle q, L_5\rangle \bigcap \langle q, L_6\rangle \bigcap H$. This is real if q, H, L_5 and L_6 are all real.

For any λ,μ,ν, the subspaces $K_5(\mu)$ and $K_6(\nu)$ are in linear general position with each of $F_4\in H_4$, $F_4(\lambda)\in H_3$, $F_4(\lambda)\in H_2$, $H_{2\,4}$, and $H_{3\,4}$. Thus, for any λ,μ,ν, the cycles $\Omega(\mathcal{T}_3)$, $\Omega_{K_4(0,\lambda)}$, $\Omega_{K_5(\mu)}$, and $\Omega_{K_6(\nu)}$ meet generically transversally, and

$$\begin{aligned}\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4(0,\lambda)}\bigcap\Omega_{K_5(\mu)}\bigcap\Omega_{K_6(\nu)} \ &= \\ &\{H_{2\,4\,6}(\mu),\ H_{2\,5\,6}(\lambda,\nu),\ H_{3\,4\,6}(\mu),\ H_{3\,5\,6}(\lambda,\nu),\ H_{4\,5\,6}\},\end{aligned} \tag{1}$$

where these five lines have coordinates, equations and arise as:

$$\begin{array}{rclr}
H_{2\,4\,6}(\mu) &=& [\,a\quad \cdot\quad \cdot\quad \mu b\quad b\,] & x_2, x_3, x_4+\mu x_5\\
H_{2\,5\,6}(\lambda,\nu) &=& [\,a\quad \cdot\quad -a\quad \nu a+\lambda b\quad b\,] & x_2, x_1+x_2+x_3, \nu x_3+x_4+\lambda x_5\\
H_{3\,4\,6}(\mu) &=& [\,\cdot\quad a\quad \cdot\quad \mu b\quad b\,] & x_1, x_3, x_4+\mu x_5\\
H_{3\,5\,6}(\lambda,\nu) &=& [\,\cdot\quad a\quad -a\quad \nu a+\lambda b\quad b\,] & x_1, x_1+x_2+x_3, \nu x_3+x_4+\lambda x_5\\
H_{4\,5\,6} &=& [\,a\quad -a\quad \cdot\quad \cdot\quad b\,] & x_1+x_2+x_3, x_3, x_4.
\end{array}$$

Thus for any λ, μ, and ν, the cycles $\Omega(\mathcal{T}_3)$, $\Omega_{K_4(0,\lambda)}$, $\Omega_{K_5(\mu)}$, and $\Omega_{K_6(\nu)}$ meet generically transversally. Then the observations 1. in $\boldsymbol{\delta,\gamma,\beta}$, and $\boldsymbol{\alpha}$ above together imply that

1. For general $\alpha,\beta,\gamma,\delta,\lambda,\mu$, and ν, the Schubert subvarieties Ω_{K_1}, Ω_{K_2}, Ω_{K_3}, Ω_{K_4}, Ω_{K_5}, and Ω_{K_6} meet generically transversally.

If λ,μ, and ν are real numbers, then $\Omega(\mathcal{T}_3)\bigcap\Omega_{K_4(0,\lambda)}\bigcap\Omega_{K_5(\mu)}\bigcap\Omega_{K_6(\nu)}$ is five real lines. Then the observations 2. in $\boldsymbol{\delta,\gamma,\beta}$, and $\boldsymbol{\alpha}$ above together imply that

2. Given any real λ,μ, and ν, for sufficiently small real numbers α,β,γ, and δ with $\gamma\neq 0$, $\Omega_{K_1}\bigcap\Omega_{K_2}\bigcap\Omega_{K_3}\bigcap\Omega_{K_4}\Omega_{K_5}\bigcap\Omega_{K_6}$ is five real lines.

Lastly, consider the cycle (1) when $(\lambda,\mu,\nu)=(0,0,0)$. We see that it equals

$$\Omega(\mathcal{T}_6) = \{H_{2\,4\,6},\ H_{2\,5\,6},\ H_{3\,4\,6},\ H_{3\,5\,6},\ H_{4\,5\,6}\}.$$

Thus $\Omega(\mathcal{T}_6)$ is in the closure of the set of degree 5 zero cycles on $\mathbf{G}_1\mathbf{P}^4$ which meet sextuples $(K_1,K_2,K_3,K_4,K_5,K_6)$ in this family.

10. Appendix II: Completion of proof of part 1 of Theorem D

In §4.3, we claimed that when $s > 0$, showing $U_{i,s}$ was a dense open subset of the product

$$\mathbf{G}^{\alpha_i - s+1}(\mathbf{P}^n/F_{l+\alpha_i-s+2}) \times \prod_{j=i+1}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$$

followed by arguments used in showing the analogous statement for $U_{i,0}$ and in the proof of Lemma 4.1. In this appendix, we give a proof.

We show that for $0 < s < \alpha_i$, $U_{i,s}$ is open and dense. Let l be the common degree of tableaux in $\mathcal{T} = \alpha_1 * \cdots * \alpha_{i-1}$. Recall that $U_{i,s} \subset \mathbf{G}^{\alpha_i-s+1}(\mathbf{P}^n/F_{l+\alpha_i-s+1}) \times \prod_{j=i+1}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ consists of those $(L,\, K_{i+1}, \ldots, K_m)$ such that the intersection

$$\left[\Omega(\mathcal{T}_{s-1,\alpha_i}; L) + \Omega(\mathcal{T}^{+s}) \bigcap \Omega_L\right] \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse and L meets $\mathcal{F}_{l,s}$ properly. Such a cycle has dimension $d = 2n - 2 - \sum_{i=1}^{m} \alpha_i$.

Let W be the open dense subset of those $(L, K_{i+1}, \ldots, K_m)$ where L meets $\mathcal{F}_{l,s}$ properly.

Suppose L meets $\mathcal{F}_{l,s}$ properly. Let $S \in \mathcal{T}_{s-1,\alpha_i}$ and $T \in \mathcal{T}^{+s}$, then $\Omega(S; L)$ is a component of $\Omega(\mathcal{T}_{s-1,\alpha_i}; L)$ and $\Omega(T) \bigcap \Omega_L$ a component of $\Omega(\mathcal{T}^{+s}) \bigcap \Omega_L$.

If $\Omega(T) \bigcap \Omega_L = \Omega(S; L)$, then $H_S \subset H_T$ so the second row of T is a subset of the second row of S, a contradiction, as $l + s$ is in the second row of T and S has degree $l + s - 1$.

Thus the singular locus Z_L of $\Omega(\mathcal{T}_{s-1,\alpha}; L) + \Omega(\mathcal{T}^{+s}) \bigcap \Omega_L$ has positive codimension. By Lemma 4.2, the set $U_L \subset W$ where $\dim(Z_L \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}) < d$ is open and dense.

Let $T \in \mathcal{T}^{+s}$ and let $\lambda = (\alpha, \beta)$ be the shape of T. Define U_T to be the set of $(L,\, K_{i+1}, \ldots, K_m) \subset W$ such that the intersection

$$\Omega(T) \bigcap \Omega_L \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse. Since $Gl_{n+1} \cdot \{F_{\alpha+1} \subset H_T\} \times \mathbf{G}^{\alpha_i-s+1}(\mathbf{P}^n/F_{l+\alpha_i-s+2}) = \mathbf{Fl}(\lambda) \times \mathbf{G}^{\alpha_i-s+1}\mathbf{P}^n$, (Regard $\mathbf{G}^{\alpha_i-s+1}(\mathbf{P}^n/F_{l+\alpha_i-s+1})$ as a subset of $\mathbf{G}^{\alpha_i-s+1}\mathbf{P}^n$) previous arguments have shown that U_T is open and dense.

Let $S \in \mathcal{T}_{s-1,\alpha_i}$ and suppose that the length of the first row of S is b. If $b \geq l + \alpha_i - s + 1$, then $\Omega(S; L) = \Omega(S)$. Let U_S consist of those $(L, K_{i+1}, \ldots, K_m)$ where the intersection

$$\Omega(S) \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse. Previous arguments have shown that U_S is open and dense.

Suppose now that $b \leq l + \alpha_i - s + 1$. Let U_S be the set of $(L, K_{i+1}, \ldots, K_m)$ where L meets $F_{b-\alpha_i+s}$ properly and the intersection

$$\Omega(S; L) \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse. Necessarily, H_S meets K_j properly for $i+1 \leq j \leq m$. Let V_S be the set where L meets $F_{b-\alpha_i+s}$ properly and for $i+1 \leq j \leq m$, H_S meets K_j properly. Then $U_S \subset V_S$. Intersecting L with $F_{b-\alpha_i+s}$ and K_j with H_S exhibits V_S as a fibre bundle over

$$\mathbf{G}^{\alpha_i-s+1}(F_{b-\alpha_i+s}/F_{l-\alpha_i+s+2}) \times \prod_{j=i+1}^{m} \mathbf{G}^{\alpha_j+1} H_S.$$

Let U'_S be the dense open set of $\mathbf{G}^{\alpha_i-s+1}(F_{b-\alpha_i+s}/F_{l-\alpha_i+s+2}) \times \prod_{j=i+1}^{m} \mathbf{G}^{\alpha_j+1} H_S$ consisting of those $(L', K'_{i+1}, \ldots, K'_m)$ for which the intersection

$$\Omega_{L'} \bigcap \Omega_{K'_{i+1}} \bigcap \cdots \bigcap \Omega_{K'_m}$$

is generically transverse in $\mathbf{G}_1 H_S$. Since $U_S \subset V_S$ is the inverse image of U'_S, we see it is a dense open subset.

Noting that $U_{i,s} = U_Z \cap \bigcap_{T\in\mathcal{T}} U_T \cap \bigcap_{S\in\mathcal{T}_{s-1,\alpha_i}} U_S$, shows that $U_{i,s}$ is a dense open subset and completes the proof of Part 1 of Theorem D. ∎

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