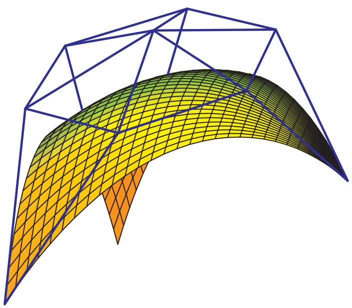
Linear precision for parametric patches

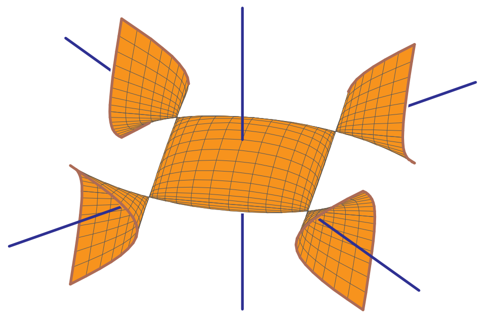
Computational Methods for Algebraic Spline Surfaces

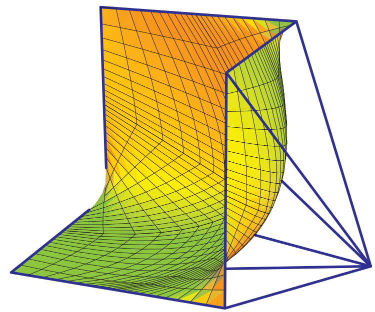
Strobl, Austria 10–14 September 2007

Frank Sottile

sottile@math.tamu.edu







Overview

(With L. Garcia, K. Ranestad, H.-C. Graf von Bothmer, and G. Craciun)

Linear precision is the ability of a patch to replicate affine functions.

- Linear precision is essentially the defining property of barycentric coordinates.
- Linear precision has an interesting mathematical formulation for toric patches.
- Any patch has a unique reparametrization having linear precision.
- This reparametrization for toric patches is computed by iterative proportional fitting, an algorithm from algebraic statistics.

Patch schemes

$\mathcal{A} \subset \mathbb{R}^d$ (e.g. $d = 2$) be a finite index set.

$\Delta :=$ convex hull of $\mathcal{A}$ (the domain).

$\beta := \{\beta_{\mathbf{a}} \colon \Delta \to \mathbb{R}_{\geq 0} \mid \mathbf{a} \in \mathcal{A}\}$, *basis functions*.

Given *control points* $b := \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ (e.g. $\ell = 3$), get

$$\varphi : \Delta \to \mathbb{R}^\ell \qquad\qquad x \longmapsto \frac{\sum \beta_{\mathbf{a}}(x)\, \mathbf{b}_{\mathbf{a}}}{\sum \beta_{\mathbf{a}}(x)}$$

The image of φ is a patch with shape Δ. $(\beta, \mathcal{A})$ is a *patch scheme*.

(Here, weights have been absorbed into the basis functions.)

Example: Bézier triangles

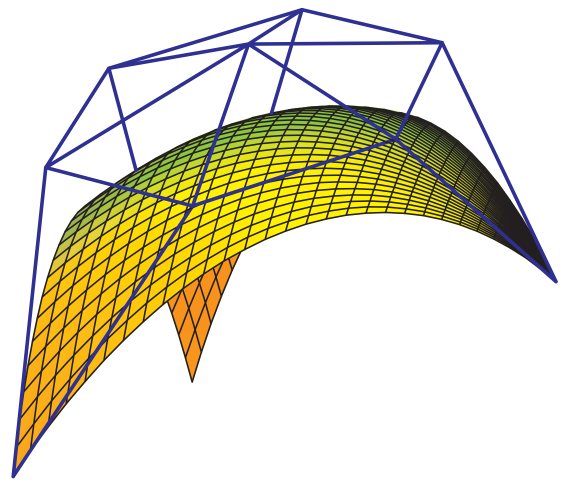
Set $\mathcal{A} := \{(\frac{i}{n}, \frac{j}{n}) \in \mathbb{N}^2 \mid i + j \leq n\}$, and

$\beta := \{\frac{n!}{i!j!(n-i-j)!} x^i y^j (1 - x - y)^{n-i-j} \mid (i, j) \in \mathcal{A}\}$.

These are the Bernstein polynomials.

Given control points, get Bézier triangle of degree n.

This picture is a cubic Bézier triangle.



Properties of path schemes

Affine invariance and the convex hull property are built into the definition.

If the control points are $\mathcal{A}$, $(\mathbf{b}_{\mathbf{a}} = \mathbf{a})$, we obtain the *tautological map*, $\tau \colon \Delta \to \Delta$,

$$x \longmapsto \frac{\sum \beta_{\mathbf{a}}(x)\, \mathbf{a}}{\sum \beta_{\mathbf{a}}(x)}$$

A patch scheme can replicate affine functions (has *linear precision*) if and only if $\tau(x) = x$. If $\mathcal{A}$ are vertices of Δ, these are *barycentric coordinates* for Δ.

$\longrightarrow$ Necessarily, τ is a homeomorphism.

Theorem. *A patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ with τ a homeomorphism has a unique reparametrization with linear precision, $\{\beta_{\mathbf{a}} \circ \tau^{-1} \mid \mathbf{a} \in \mathcal{A}\}$.*

Geometric formulation of a patch

The unique reparametrization having linear precision is rarely rational.

Let $\mathbb{P}^{\mathcal{A}}$ be the projective space with coordinates $[y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$, and set

$$\beta : \Delta \to \mathbb{P}^{\mathcal{A}} \qquad x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}] \,.$$

Define $X_\beta :=$ image of β. This is independent of reparametrization.

Control points correspond to linear projections.

Given $b = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$, let

$$\pi_{\mathbf{b}} : \mathbb{P}^{\mathcal{A}} -\!\!\to \mathbb{R}^\ell \subset \mathbb{P}^\ell \qquad y \longmapsto \frac{\sum y_{\mathbf{a}} \mathbf{b}_{\mathbf{a}}}{\sum y_{\mathbf{a}}}$$

Then the map given by $(\beta, \mathcal{A})$ with control points b is $\pi_b \circ \beta$.

Geometry of rational linear precision

Let $\overline{X_\beta}$ be the Zariski closure of X_β in $\mathbb{CP}^{\mathcal{A}}$.

Theorem. *The reparametrization of a patch X_β with linear precision is given by rational functions if and only if the map*

$$\pi_{\mathcal{A}} : \overline{X_\beta} \dashrightarrow \mathbb{CP}^d \qquad y \longmapsto \frac{\sum y_{\mathbf{a}}\, \mathbf{a}}{\sum y_{\mathbf{a}}} \in \mathbb{C}^d \subset \mathbb{CP}^d$$

has degree 1.

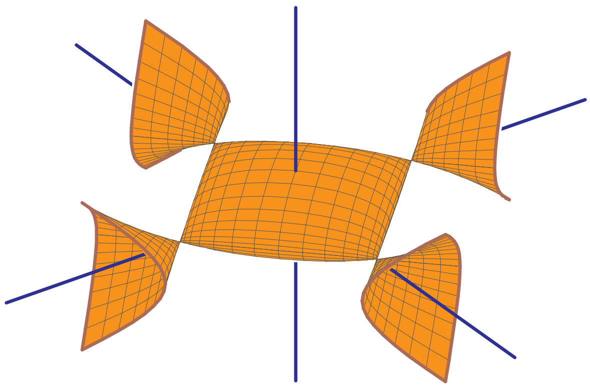
When this happens, the patch has *rational linear precision*.

Toric patches

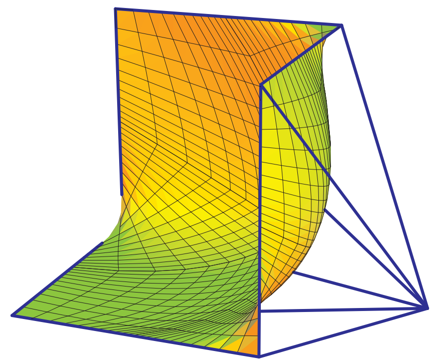
Krasauskas defined *toric patches* X_Δ, which are a class of multi-sided patches that generalize Bézier patches.

For these, $\mathcal{A}$ is the set of integer points in polytope Δ.

Basis functions are a natural generalization of the Bernstein polynomials.



Toric Variety (projected)



Toric Patch

Rational linear precision for toric patches

To study a toric patch X_Δ up to reparametrization, can assume

$\beta := \{c_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ where $c_{\mathbf{a}} > 0$, and

$\mathbb{R}^d_{>0}$:= the domain.

Set $f := \sum c_{\mathbf{a}} x^{\mathbf{a}}$.

Theorem. X_Δ *has rational linear precision if and only if the rational map*

$$x \longmapsto \frac{1}{f}\left(x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \ldots, x_n \frac{\partial f}{\partial x_n}\right)$$

is a birational isomorphism $\mathbb{C}^d - \rightarrow \mathbb{C}^d$.

Iterative proportional fitting

The condition of the theorem is very restrictive, and it is likely that rational linear precision is possible only for the classical patches. (Ranestad, Graf van Bothmer, and I begin to treat this for surfaces $d = 2$).

Despite not being rational, the reparametrization of a toric patch having linear precision may be computed using a fast iterative numerical algorithm from statistics, called iterative proportional fitting (IPF).

In fact, there is an interesting dictionary between concepts of toric patches from geometric modeling and exponential families form statistics.