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New fewnomial upper bounds

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1 Introduction

Unlike complex solutions, the number of real solutions to a system of polynomials depends upon the number of monomials and not on the degree.

In 1980, Askold Khovanskii [3] showed that there are at most

$$2^{\binom{n+k}{2}}(n+1)^{n+k}$$

nondegenerate positive real solutions to a system of Laurent polynomials

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0 \qquad (1)$$

having a total of $n+k+1$ distinct monomials.

The fewnomial bound asserts that a system of 2 polynomials in 2 variables with 5 monomials ($k=n=2$) has at most 5184 positive solutions. The first concrete evidence that it was overstated was given by Li, Rojas, Wang in 2002 [4], who showed that 2 trinomials in 2 variables have at most 5 positive solutions. This is a (non-general) system with $n=k=2$.

2 New Fewnomial Bound

Theorem. A system (1) of n Laurent polynomials in n variables having a total of $n+k+1$ monomials has fewer than

$$\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$$

positive solutions [1]. For k fixed, this is asymptotically sharp in n [2].

The proof uses a completely different approach than Khovanskii's induction. When $n=k=2$, this bound is 20 $(\ll 5184)$!

3 Gale Dual System

Typically, a polynomial system (1) is formulated as an intersection

$$X \;\cap\; L \;\subset\; \mathbb{P}^{n+k}$$

X : The toric variety parametrized by the monomials in (1)

$L \simeq \mathbb{P}^k$: A linear subspace (which depends upon the coefficients of the f_i)

We may instead parametrize L by $n+k$ linear polynomials $p_i(y)$ for $y \in \mathbb{R}^k$, obtaining an equivalent *Gale system*

$$\prod_{i=1}^{n+k} p_i(y)^{a_{i,j}} \;=\; 1 \qquad \text{for} \quad j=1,\ldots,k \qquad (2)$$

(In the torus of $\mathbb{P}^{n+k}$, the toric variety is defined by $\prod_{i=1}^{n+k} z_i^{a_{i,j}} = 1$ for $j=1,\ldots,k$.)

For example, the system

$$\begin{aligned} t^{-1183}u^{691}z^5w^5 &= 3-t^{492} \\ t^{-955}u^{463}z^5w^5 &= 3-u^{492} \\ z &= \sqrt[60]{35}(4-2t^{492}+u^{492}) \\ w &= \sqrt[30]{10}(1+2t^{492}-u^{492}) \end{aligned}$$

becomes

$$\begin{aligned} F_1(x,y) &:= \frac{3500^{12}x^{27}(3-x)^8(3-y)^8}{y^{15}(4-2x+y)^{60}(1+2x-y)^{60}} = 1 \\ F_2(x,y) &:= \frac{3500^{12}x^8y^4(3-y)^{45}}{(3-x)^{33}(4-2x+y)^{60}(1+2x-y)^{60}} = 1 \end{aligned} \qquad (3)$$

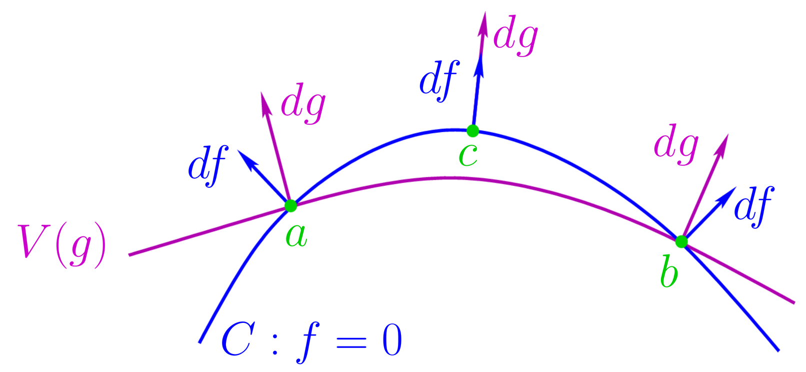
Theorem. The system (1) of polynomials on $\mathbb{R}^n_>$ is equivalent to the Gale system (2) of rational functions on the polyhedron

$$\Delta \;:=\; \{y \in \mathbb{R}^k \;\mid\; p_i(y) > 0 \quad \text{for} \quad i=1,\ldots,n{+}k\}\,. \qquad (4)$$

4 Khovanskii-Rolle Theorem

The idea of the proof is to transform the difficult problem of solving the high degree equations (2) to solving relatively low degree polynomial systems.

Our tool is the Khovanskii-Rolle Theorem: Between any two points where $g=0$ along an arc of a curve C, there is a point where the differential dg is normal to the curve. We illustrate this in 2 dimensions.



$df \wedge dg(a) > 0$, $df \wedge dg(c) = 0$, and $df \wedge dg(b) < 0$.

Suppose C_k is defined by $f_1 = \ldots = f_{k-1} = 0$ and $g = f_k$. the Jacobian $J_k = \text{Jac}(f_1,\ldots,f_k)$ vanishes where dg is normal to the curve C_k. If $\flat(C_k)$ is the number of non-compact components of C_k, and we write $|f_1,\ldots,f_k|$ for the number of common zeroes in Δ to the f_i, then

$$|f_1,\ldots,f_k| \;\le\; \flat(C_k) \;+\; |f_1,\ldots,f_{k-1},J_k|\,. \qquad (5)$$

5 Sketch

We replace the rational equations (2) by the transcendental equations

$$f_j \;:=\; \sum_{i=1}^{n+k} a_{i,j}\log(p_i(y)) \;=\; 0 \qquad j=1,\ldots,k$$

The two systems have the same zeroes in Δ (4). The advantage is that the Jacobian J_k is a polynomial of degreee n.

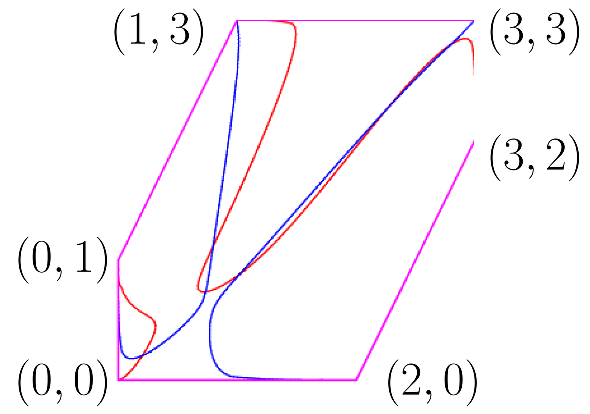
To iterate the Khovanskii-Rolle Theorem, define successive Jacobians $J_i := \text{Jac}(f_1,\ldots,f_i,J_{i+1},\ldots,J_k)$, which are polynomials of degree $2^{k-i}n$ and curves C_i by $f_1 = \cdots = f_{i-1} = J_{i+1} = \cdots = J_k$. Then

$$|f_1,\ldots,f_k| \;\le\; \flat(C_k) \;+\; \cdots \;+\; \flat(C_1) \;+\; |J_1,\ldots,J_k|\,.$$

Estimating the right hand side gives the new fewnomial bound $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$.

6 Example

The system (3) asks for the 6 intersection points of the two curves below



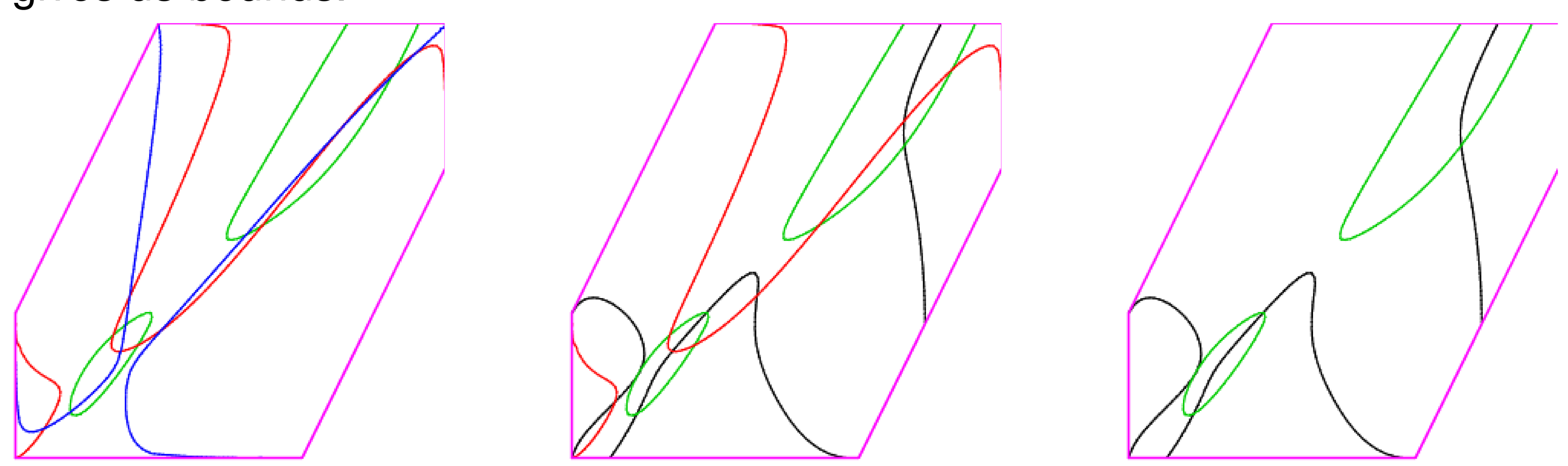
$$\begin{aligned} p_1(x,y) &: & x &\ge 0 \\ p_2(x,y) &: & y &\ge 0 \\ p_3(x,y) &: & 4-2x+y &\ge 0 \\ p_4(x,y) &: & 2x-y+1 &\ge 0 \\ p_5(x,y) &: & 3-x &\ge 0 \\ p_6(x,y) &: & 3-y &\ge 0 \end{aligned}$$

Here are the Jacobians

$$\begin{aligned} \Gamma_2 \;:=\; & 2736 - 15476x + 2564y + 32874x^2 - 21075xy + 6969y^2 \\ & -10060x^3 - 7576x^2y + 8041xy^2 - 869y^3 \;+\; 7680x^3y - 7680x^2y^2 + 1920xy^3\,. \end{aligned}$$

$$\begin{aligned} \Gamma_1 \;:=\; & 8357040x - 2492208y - 25754040x^2 + 4129596xy - 10847844y^2 \\ & - 37659600x^3 + 164344612x^2y - 65490898xy^2 + 17210718y^3 \;+ 75054960x^4 \\ & - 249192492x^3y + 55060800x^2y^2 + 16767555xy^3 - 2952855y^4 \;- 36280440x^5 \\ & +143877620x^4y + 35420786x^3y^2 - 80032121x^2y^3 + 19035805xy^4 - 1128978y^5 \;+ 5432400x^6 \\ & -33799848x^5y - 62600532x^4y^2 + 71422518x^3y^3 - 13347072x^2y^4 - 1836633xy^5 + 211167y^6 \\ & +2358480x^6y + 21170832x^5y^2 - 13447848x^4y^3 - 8858976x^3y^4 + 7622421x^2y^5 - 1312365xy^6 \\ & - 1597440x^6y^2 - 1228800x^5y^3 + 4239360x^4y^4 - 2519040x^3y^5 + 453120x^2y^6\,. \end{aligned}$$

Their solutions intersperse the solutions to $\Gamma_2 = F_1 = 0$ in the hexagon, which gives us bounds.



$\Gamma_2 = 0,\ F_1 = 0,\ F_2 = 0$ $\Gamma_1 = 0,\ \Gamma_2 = 0,\ F_1 = 0$ $\Gamma_1 = 0,\ \Gamma_2 = 0$

$$\begin{array}{ccccccccccc} |F_1,F_2| & \le & \flat(F_1=0) & + & |F_1,\Gamma_2| & \le & \flat(F_1=0) & + & \flat(\Gamma_2=0) & + & |\Gamma_1,\Gamma_2| \\ 6 & \le & 2 & + & 4 & \le & 2 & + & 1 & + & 5 \end{array}$$

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[4] Tien-Yien Li, J. Maurice Rojas, and Xiaoshen Wang, *Counting real connected components of trinomial curve intersections and m-nomial hypersurfaces*, Discrete Comput. Geom. **30** (2003), no. 3, 379–414.