Gale duality for complete intersections

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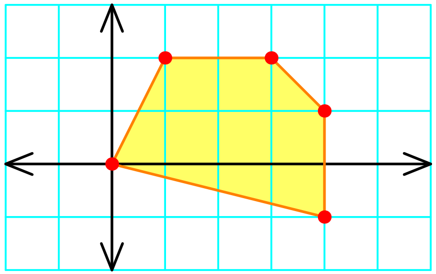
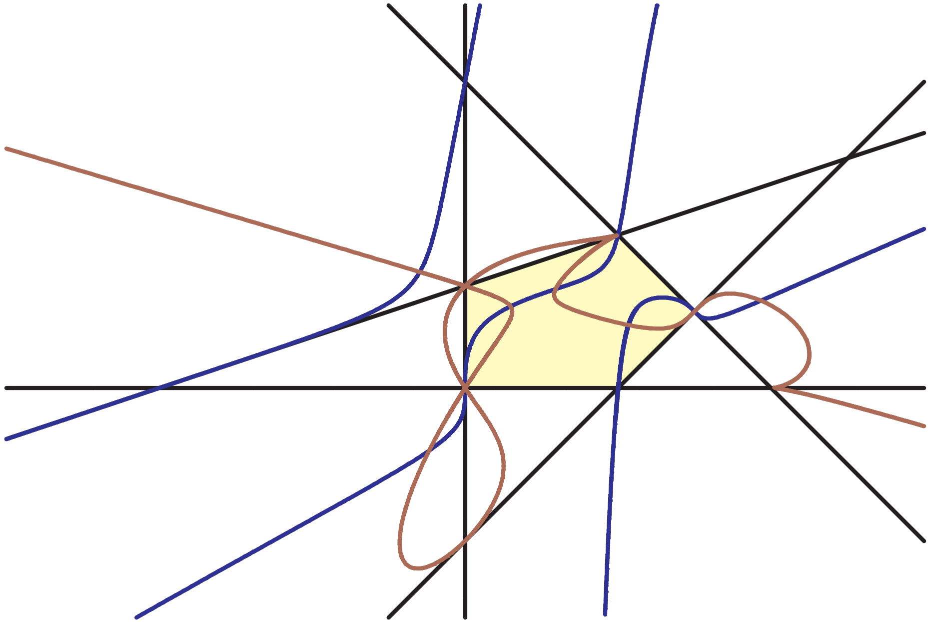




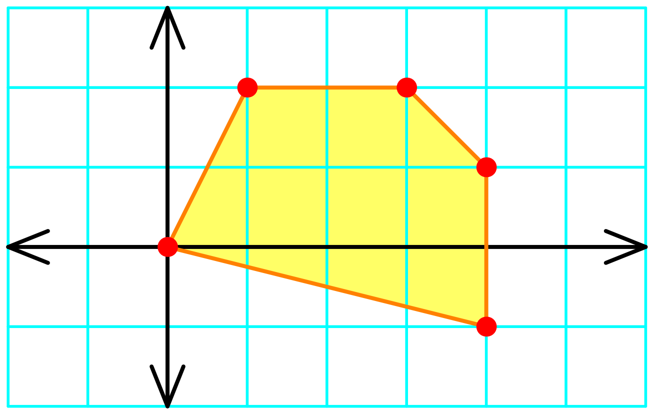
Gale duality

Gale duality for complete intersections asserts that systems of n polynomial equations in $m+n$ variables are equivalent to certain systems of l rational functions in $l+m$ variables. This allows us to conclude that

$$\frac{(2x-3y)^2(4x+y-7)^3}{(1+x-3y)^2(x-7y-2)} \;=\; \frac{(2x-3y)(x-7y-2)^3}{(1+x-3y)^3(4x+y-7)} \;=\; 1\,.$$

has 17 solutions where $(4x+y-7)(x-7y-2)(1+x-3y)(2x-3y) \neq 0$.

This is because the pentagon at right (whose vertices annihilate the exponents in the equations) has area $17/2$.



Master Functions

Let $\mathcal{H}$ be an essential arrangement of hyperplanes in $\mathbb{C}^{l+m}$ defined by affine functions $p_1(y), \ldots, p_{l+m+n}(y)$.

A *weight* for $\mathcal{H}$ is a vector $\beta = (b_1, \ldots, b_{l+m+n}) \in \mathbb{Z}^{l+m+n}$ of integers. This defines the *master function* for $\mathcal{H}$ with weight β

$$p(y)^{\beta} \;:=\; p_1(y)^{b_1} \cdot p_2(y)^{b_2} \cdots p_{l+m+n}(y)^{b_{l+m+n}}\,,$$

which is a rational function defined on the complement $M_{\mathcal{H}}$ of the arrangement.

A *master function complete intersection with weights* $\mathcal{B} = (\beta_1, \ldots, \beta_l)$ is a subscheme of $M_{\mathcal{H}}$ of dimension m which may be defined by a system of master functions

$$p(y)^{\beta_1} \;=\; p(y)^{\beta_2} \;=\; \cdots \;=\; p(y)^{\beta_l} \;=\; 1\,.$$

NB: The weights $\mathcal{B}$ are necessarily linearly independent.

Sparse polynomials

Let $\mathcal{A} = \{0, \alpha_1, \ldots, \alpha_{l+m+n}\} \subset \mathbb{Z}^{m+n}$ be integer vectors which are exponents for Laurent monomials in $x_1, \ldots, x_{m+n}$. A *sparse polynomial* f with *support* $\mathcal{A}$ is a polynomial whose monomials are $1, x^{\alpha_1}, \ldots, x^{\alpha_{l+m+n}}$. Because the exponents can be negative, f is a function on the algebraic torus, $(\mathbb{C}^\times)^{m+n}$.

A complete intersection with support $\mathcal{A}$ is a subscheme of $(\mathbb{C}^\times)^{m+n}$ of dimension m which may be defined by a system of polynomials,

$$f_1(x_1, \ldots, x_{m+n}) \;=\; f_2(x_1, \ldots, x_{m+n}) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_{m+n}) \;=\; 0\,,$$

here each polynomial f_i has support $\mathcal{A}$.

These are well-studied algebraic sets, but are in fact no different than master function complete intersections.

The geometry of master functions

The affine functions $p_1(y), \ldots, p_{l+m+n}(y)$ define an injective map

$$\psi_p \ : \ \mathbb{C}^{l+m} \ \longrightarrow \ \mathbb{C}^{l+m+n} \qquad\qquad (\text{set } L := \psi_p(\mathbb{C}^{l+m}))$$

and the hyperplane complement $M_{\mathcal{H}}$ is $\psi_p^{-1}((\mathbb{C}^\times)^{l+m+n})$.

The weights $\mathcal{B} = (\beta_1, \ldots, \beta_l)$ define a subtorus of $(\mathbb{C}^\times)^{l+m+n}$

$$\mathbb{T} \ := \ \{z \in (\mathbb{C}^\times)^{l+m+n} \mid z^{\beta_1} = z^{\beta_2} = \cdots = z^{\beta_l} = 1\}\,,$$

which is connected if and only if $\mathcal{B}$ is saturated ($\mathbb{Z}\mathcal{B} = \mathbb{Q}\mathcal{B} \cap \mathbb{Z}^{l+m+n}$).

In this way, the system of master functions

$$p(y)^{\beta_1} \ = \ p(y)^{\beta_2} \ = \ \cdots \ = \ p(y)^{\beta_l} \ = \ 1\,.$$

equals $\psi_p^{-1}(\mathbb{T})$, which is isomorphic to $\mathbb{T} \cap L$.

The geometry of sparse polynomials

The map $\varphi_{\mathcal{A}}\colon (\mathbb{C}^\times)^{m+n} \ni x \longmapsto (x^{\alpha_1}, \ldots, x^{\alpha_{l+m+n}}) \in (\mathbb{C}^\times)^{l+m+n}$ pulls an affine function $\Lambda := c_0 + \sum_i c_i z_i$ on $\mathbb{C}^{l+m+n}$ back to a sparse polynomial

$$\varphi_{\mathcal{A}}^*(\Lambda) \;=\; c_0 + \sum_{i=1}^{l+m+n} c_i x^{\alpha_i}$$

with support $\mathcal{A}$.

In this way, a system of sparse polynomials $f_1 = \cdots = f_n$ is the pullback of a system of affine functions $\Lambda_1 = \cdots = \Lambda_n$ on $\mathbb{C}^{l+m+n}$. These define an affine subspace L of $\mathbb{C}^{l+m+n}$ of dimension $l+m$ and the system equals $\varphi_{\mathcal{A}}^{-1}(L)$.

When $\mathbb{Z}\mathcal{A} = \mathbb{Z}^{m+n}$ ($\mathcal{A}$ is primitive), $\varphi_{\mathcal{A}}$ is injective. Set $\mathbb{T} := \varphi_{\mathcal{A}}(\mathbb{C}^\times)^{m+n}$. Then the system $\varphi_{\mathcal{A}}^{-1}(L)$ is isomorphic to $\mathbb{T} \cap L$.

Gale duality

The master function complete intersection with exponents $\mathcal{B}$ is isomorphic to the complete intersection with support $\mathcal{A}$ when

$$\text{(Master function)} \qquad \mathbb{T} \cap L \;=\; \mathbb{T} \cap L \qquad \text{(Sparse polynomial)}\,.$$

Unpacking the definitions, we get

Theorem. *Suppose that $\mathcal{A}$ is primitive, $\mathcal{B}$ is saturated, $\Lambda_1, \ldots, \Lambda_n$ define the sparse polynomial system, and $p_1(y), \ldots, p_{l+m+n}(y)$ define $\mathcal{H}$. If*

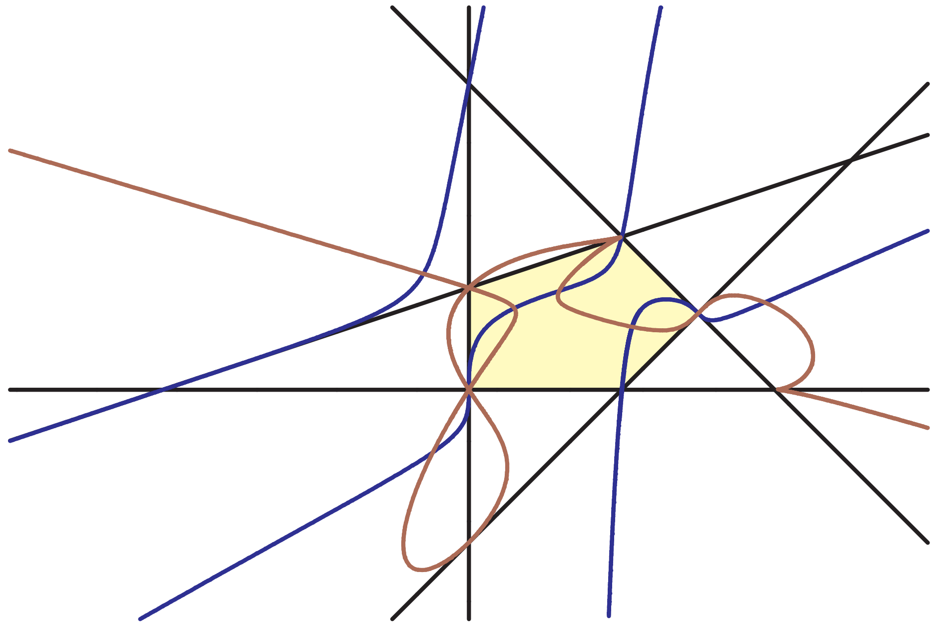
- $\Lambda_1 = \cdots = \Lambda_n$ *defines the linear subspace* $L = \psi_p(\mathbb{C}^{l+m})$, *and*
- $\mathcal{A} \cdot \mathcal{B} = 0$, *where the matrix $\mathcal{A}$ has column vectors α_i and $\mathcal{B}$ has column vectors β,*

then the master function complete intersection is isomorphic to the complete intersection with support $\mathcal{A}$.

An Example

$$\frac{x^2(1-x-y)^3}{y^2(\frac{1}{2}-x+y)\left(\frac{10}{11}(1+x-3y)\right)^2} \;=\; \frac{y^3(1-x-y)}{x(\frac{1}{2}-x+y)^3\left(\frac{10}{11}(1+x-3y)\right)} \;=\; 1\,,$$

defines a 0-dimensional set in the complement of the lines defined by the linear factors.
(We drew the curves.)



Example Continued

If we order the affine functions,

$$x\,,\;\; y\,,\;\; (1-x-y)\,,\;\; (\tfrac{1}{2}-x+y)\,,\;\; \tfrac{10}{11}(1+x-3y)\,,$$

our master functions

$$\frac{x^2(1-x-y)^3}{y^2(\frac{1}{2}-x+y)\left(\frac{10}{11}(1+x-3y)\right)^2} \quad \text{and} \quad \frac{y^3(1-x-y)}{x(\frac{1}{2}-x+y)^3\left(\frac{10}{11}(1+x-3y)\right)}$$

have exponents $(2,-2,3,-1,-2)$ and $(-1,3,1,-3,-1)$.

Observe that

$$\begin{aligned}
\left(u^2v\right)^2 \cdot \left(uv^2w\right)^{-2} \cdot \left(v^2w^3\right)^3 \cdot \left(v^2w\right)^{-1} \cdot \left(uvw^3\right)^{-2} &= 1, \quad \text{and}\\
\left(u^2v\right)^{-1} \cdot \left(uv^2w\right)^{3} \cdot \left(v^2w^3\right) \cdot \left(v^2w\right)^{-3} \cdot \left(uvw^3\right)^{-1} &= 1\,.
\end{aligned}$$

Example completed

Because we have

$$
\begin{aligned}
(u^2v)^2 \cdot (uv^2w)^{-2} \cdot (v^2w^3)^3 \cdot (v^2w)^{-1} \cdot (uvw^3)^{-2} &= 1, \quad \text{and} \\
(u^2v)^{-1} \cdot (uv^2w)^3 \cdot (v^2w^3) \cdot (v^2w)^{-3} \cdot (uvw^3)^{-1} &= 1\,.
\end{aligned}
$$

if we substitute u^2v for x, uv^2w for y, and the corresponding affine functions for the last three monomials, we get the system

$$
\begin{aligned}
v^2w^3 &= 1 - x - y = 1 - u^2v - uv^2w \\
v^2w &= \tfrac{1}{2} - x + y = \tfrac{1}{2} - u^2v + uv^2w \\
uvw^3 &= \tfrac{10}{11}(1 + x - 3y) = \tfrac{10}{11}(1 + u^2v - 3v^2w^3)
\end{aligned}
$$

whose solutions are isomorphic to the solutions to the system of master functions.