Linear precision for parametric patches

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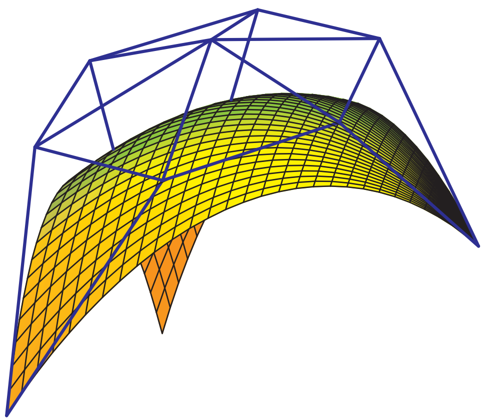
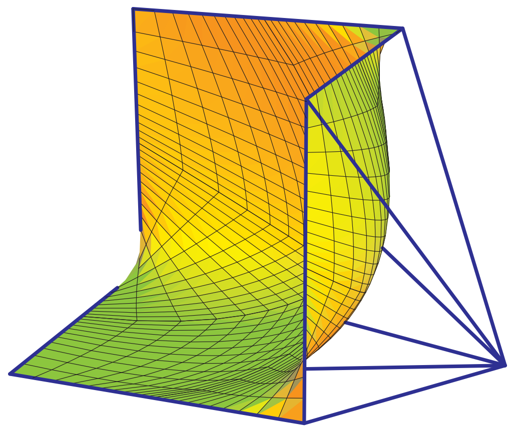




Overview

(With L. Garcia, K. Ranestad, and H.C. Graf v. Bothmer)

Linear precision, the ability of a patch to replicate affine functions, has interesting properties and connections to other areas of mathematics.

- Any patch has a unique reparametrization (possibly non-rational) with linear precision. This reparametrization is the maximum likelihood estimator from algebraic statistics.
- This reparametrization for toric patches is computed by iterative proportional fitting, an algorithm from statistics.
- Linear precision has an interesting mathematical formulation for toric patches, which leads to a classification, using algebraic geometry, of toric surface patches having linear precision.

(Control-point) patch schemes

Let $\mathcal{A} \subset \mathbb{R}^d$ (e.g. $d = 2$) be a finite set with convex hull Δ, and $\beta := \{\beta_{\mathbf{a}} \colon \Delta \to \mathbb{R}_{\geq 0} | \mathbf{a} \in \mathcal{A}\}$, *basis functions* with $1 = \sum_{\mathbf{a}} \beta_{\mathbf{a}}(x)$.

Given *control points* $\{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ (e.g. $\ell = 3$), get a map

$$\varphi \ : \ \Delta \longrightarrow \mathbb{R}^\ell \qquad\qquad x \longmapsto \sum \beta_{\mathbf{a}}(x)\, \mathbf{b}_{\mathbf{a}}$$

Image of φ is a patch with shape Δ. Call $(\beta, \mathcal{A})$ is a *patch*.

Affine invariance and the convex hull property are built into definition.

Linear precision is the ability to replicate linear functions.

We will adopt a precise, but restrictive definition.

Linear Precision

Let $\mathcal{A}$ be the control points, $(\mathbf{b_a} = \mathbf{a})$, to get the *tautological map*,

$$\tau \;:\; x \longmapsto \sum \beta_{\mathbf{a}}(x)\,\mathbf{a} \qquad \tau \;:\; \Delta \to \Delta\,.$$

$(\beta, \mathcal{A})$ has *linear precision* if and only if $\tau =$ identity map.

Theorem (G-S). *If τ is a homeomorphism, the patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has a unique reparametrization with linear precision, $\{\beta_{\mathbf{a}} \circ \tau^{-1} \mid \mathbf{a} \in \mathcal{A}\}$.*

How to compute τ^{-1}? The map τ factors

$$\begin{array}{rcccl} \varphi\colon \Delta & \xrightarrow{\beta} & \mathbb{RP}^{\mathcal{A}} & \xrightarrow{\mu} & \Delta \\ x & \longmapsto & [1, \beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}] \qquad [0, \ldots, 1, \ldots, 0] & \longmapsto & \mathbf{a} \end{array}$$

Note $\beta \circ \tau^{-1} = \mu^{-1}\colon \Delta \to X_\beta$, where $X_\beta :=$ image $\beta(\Delta) \subset \mathbb{RP}^{\mathcal{A}}$.

We shall see that μ^{-1} is the key.

Toric patches (After Krasauskas)

A polyotope Δ with integer vertices is given by facet inequalities

$$\Delta \;=\; \{x \in \mathbb{R}^d \mid h_i(x) \geq 0 \ i = 1, \ldots, n\}\,,$$

where h_i is linear with integer coefficients.

For each $\mathbf{a} \in \mathcal{A} := \Delta \cap \mathbb{Z}^d$, there is a *toric Bézier function*

$$\beta_{\mathbf{a}}(x) \;:=\; h_1(x)^{h_1(ba)} \cdots h_n(x)^{h_n(\mathbf{a})}\,.$$

Let $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}_>$ be positive weights. The *toric patch* $(w, \mathcal{A})$ has blending functions $\{w_{\mathbf{a}}\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$. Write $X_{w,\mathcal{A}}$ for its image in $\mathbb{RP}^{\mathcal{A}}$, which is the positive part of a toric variety.

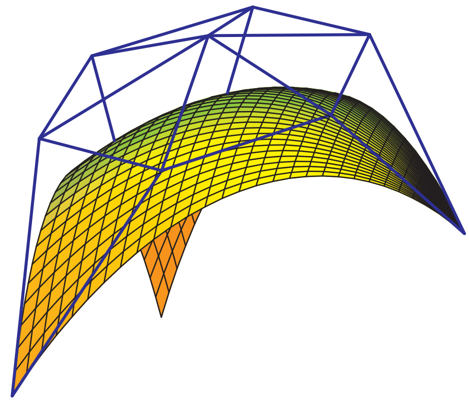
The map $\mu\colon X_{w,\mathcal{A}} \to \Delta$ is the *algebraic moment map*.

Example: Bézier triangles

Bézier triangles are toric surface patches.

Set $\mathcal{A} := \{(i,j) \in \mathbb{N}^2 \mid i \geq 0,\ j \geq 0, n-i-j \geq 0\}$, then
$w_{(i,j)}\beta_{(i,j)} := \frac{n!}{i!j!(n-i-j)!}x^i y^j (n-x-y)^{n-i-j}$.

These are essentially the Bernstein polynomials, which have linear precision.
The corresponding toric variety is the Veronese surface of degree n.
Choosing control points, get Bézier triangle of degree n.
This picture is a cubic Bézier triangle.



Digression: algebraic statitics

In algebraic staitstics, the probability simplex $= \mathbb{RP}^n_>$, the positive part of $\mathbb{RP}^n$, and its subvarieties $X_{w,\mathcal{A}}$ are called *toric statistical models*.

For example, the subvariety corresponding to the Bézier triangle is the *trinomial distribution*.

The algebraic moment map $\mu\colon \mathbb{RP}^n_> \longrightarrow \Delta$ is called the *expectation map*, and, for $p \in \mathbb{RP}^n_>$, the point $\mu^{-1}(\mu(p)) \in X_{w,\mathcal{A}}$ is the *maximum likelihood estimator*, the distribution in $X_{w,\mathcal{A}}$ which 'best' explains p.

Iterative proportional fitting (IPF) is a fast numerical algorithm to compute μ^{-1}. IPF may be useful in modeling.

Linear precision means maximum likelihood degree 1. Many statistical models have MLD 1.

Linear precision for toric patches

Given the data $(w, \mathcal{A})$ of a toric patch, define a polynomial

$$F_{w,\mathcal{A}} := \sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}},$$

where $x^{\mathbf{a}}$ is the multivariate monomial.

Theorem (G-S). *A toric patch* $(w, \mathcal{A})$ *has linear precision if and only if*

$$\mathbb{C}^d \ni x \longmapsto \left(x_1 \frac{\partial F_{w,\mathcal{A}}}{\partial x_1},\ x_2 \frac{\partial F_{w,\mathcal{A}}}{\partial x_2},\ \ldots,\ x_d \frac{\partial F_{w,\mathcal{A}}}{\partial x_d}\right) \qquad (*)$$

defines a birational isomorphism $\mathbb{C}^d - \to \mathbb{C}^d$.

We say that F *defines a toric polar Cremona transformation*, when its toric derivatives $(*)$ define a birational map.

Linear precision for toric surface patches

Theorem (GvB-R-S). *A polynomial* $F \in \mathbb{C}[x, y]$ *defines a toric polar Cremona transformation if and only if it is equivalent to one of the following forms*

- $(x + y + 1)^n$ *($\Longleftrightarrow$ Bézier triangle).*
- $(x + 1)^m (y + 1)^n$ *($\Longleftrightarrow$ tensor-product patch).*
- $(x + 1)^m \big((x + 1)^d + y\big)^n$ *($\Longleftrightarrow$ trapezoidal patch).*
- $(x^2 + y^2 + z^2 - 2(xy + xz + yz))^d$. *(no analog in modeling).*

In particular, this classifies toric surface patches that enjoy linear precision.

Ideas in proof

Using the classification of birational maps $\mathbb{P}^2 - \to \mathbb{P}^2$ and that F lies in the linear series of toric polar derivatives, we

- Restrict the singularities of $F = 0$ to at most one node outside the axes, in which case F is contracted.
- Conclude that $F = 0$ is rational and then study/use parametrizations $\mathbb{P}^1 \to \{F = 0\}$.
- Finish it up wth some local calculations.

- Almost none of these techniques are available in higher dimensions.

Bibliography

- Rimvydas Krasauskas, *Toric surface patches*, Adv. Comput. Math. **17** (2002), no. 1-2, 89–133.
- Luis Garcia-Puente and Frank Sottile, *Linear precision for parametric patches*, 2007, `ArXiV:0706.2116`.
- Hans-Christian Graf van Bothmer, Kristian Ranestad, and Frank Sottile, *Linear precision for toric surface patches*, 2008. `ArXiv:0806.3230`.

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Future work?

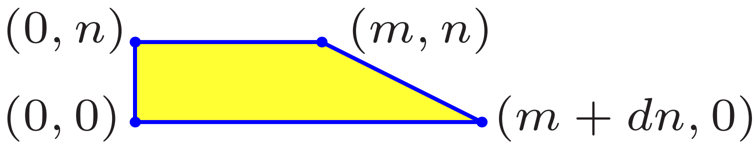
- When is it possible to tune a patch (move the points $\mathcal{A}$) to acheive linear precision?
- Linear precision for 3- and higher-dimensional patches.
- Algebraic statistics furnishes many higher dimensional toric patches with linear precision.
- Can iterative proportional fitting be useful to compute patches?

Trapezoidal patch

Let $n, d \geq 1$ and $m \geq 0$ be integers, and set

$$\mathcal{A} \;:=\; \{(i,j) \;:\; 0 \leq j \leq n \;\text{ and }\; 0 \leq i \leq m + dn - dj\}\,,$$

which are the integer points inside the trapezoid below.



Choose weights $w_{i,j} := \binom{n}{j}\binom{m+dn-dj}{i}$.
Then the toric Bézier functions are

$$\binom{n}{j}\binom{m+dn-dj}{i} s^i (m + dn - s - dt)^{m+dn-dj-i} t^j (n - t)^{n-j}\,.$$