Bounds for real solutions to structured polynomial systems

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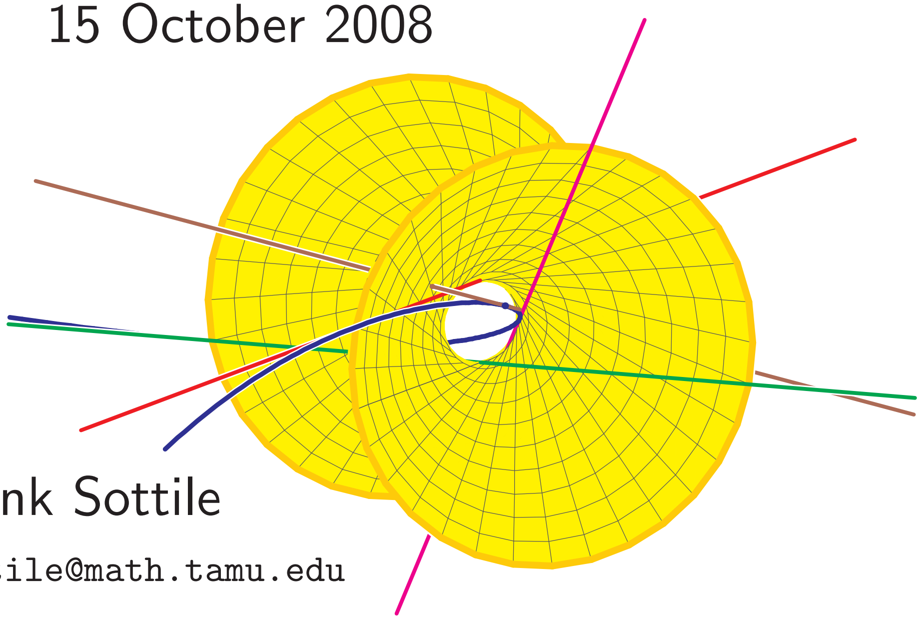


Frank Sottile

sottile@math.tamu.edu

Bounds for real solutions

Given a system of polynomials

$$f_1(x_2,\ldots,x_n) \;=\; \cdots \;=\; f_N(x_1,\ldots,x_n) \;=\; 0\,,$$

with d nondegenerate complex solutions, what can we say about its number, r, of real solutions, (besides the trivial

$$d \;\geq\; r \;\geq\; d \mod 2 \in \{0,1\}\;?\;)$$

While the answer in general is *nothing*, when the equations have special structure coming from geometry, we can often say a great deal about r, or the positive solutions, r_+.

- Sometimes, there is a smaller, sharp upper bound than d
- Often, there is a lower bound larger than $d \mod 2$
- In some cases the lower bound is d.

Complex bounds for sparse systems

An integer vector $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n$
corresponds to a (Laurent) monomial, $x^\alpha := x_1^{a_1} \ldots x_n^{a_n}$.

A polynomial with support $\mathcal{A} \subset \mathbb{Z}^n$ is

$$f \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha \qquad c_\alpha \in \mathbb{R}(\text{ or } \in \mathbb{C})\,.$$

Kushnirenko's Theorem. *A general system of polynomials*

$$f_1(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0$$

with support $f_i = \mathcal{A}$ *has* $d = n!\mathrm{vol}(\mathrm{conv}(\mathcal{A}))$ *solutions.*

Bernstein's Theorem. *If the polynomials have different supports,* $\mathcal{A}_1, \ldots, \mathcal{A}_n$, *then* $d =$ *mixed volume of* $\mathrm{conv}(\mathcal{A}_i)$, $i = 1, \ldots, n$.

Upper bounds

By Descartes' rule of signs (*c.* 1637),

$$c_0 x^{a_0} + c_1 x^{a_1} + \cdots + c_n x^{a_n} = 0$$

has $r_+ \leq n$ ($\leq n$ positive solutions).

Khovanskii (*c.* 1980) gave a multivariate generalization.

Theorem. *A system of polynomial equations*

$$f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0$$

with $1 + l + n$ *monomials (e.g.* $|\mathcal{A}| = 1 + l + n$*) has*

$$r_+ < 2^{\binom{l+n}{2}} (n+1)^{l+n} .$$

$\Rightarrow r_+$ has a completely different character than d.

New upper bounds

Khovanski more generally gave a topological method to bound solutions to systems of equations. Significant improvements to his bound have recently been found that take advantage of some (simple) geometry and combinatorics available for systems of polynomials.

Given a system with support $\mathcal{A}$ where $|\mathcal{A}| = 1+l+n$,

$$f_1(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0\,.$$

Theorem (Bihan-S.). *We have*

$$r_+ \;<\; \sum_{j=1}^{l} 2^{\binom{l-j}{2}} n^{l-j} \binom{n+l+1}{j} \;<\; \tfrac{e^2+3}{4} 2^{\binom{l}{2}} n^l\,.$$

Theorem (Bates-Bihan-S.). *When $\mathcal{A}$ is primitive,* $r < \frac{e^4+3}{4} 2^{\binom{l}{2}} n^l$.

Theorem (Bihan-Rojas-S.). *These are sharp for l fixed and $n \gg 0$.*

Three challenges

The fewnomial bounds

$$2^{\binom{l+n}{2}}(n+1)^{l+n} \qquad \text{and} \qquad \frac{e^2+3}{4}2^{\binom{l}{2}}n^l$$

are both exponential in l^2.

Challenge 1. Find a bound with lower order in l.

It is easy to construct systems with $O(l^n)$ real solutions.

Challenge 2. Construct systems with more real soutions.

These results are for systems of polynomials with the same supports.

Challenge 3. Give bounds for polynomials with different supports.

Lower bounds

Many geometric problems enjoy a non-trivial lower bound on their number of real solutions, which is an existence proof for real solutions.

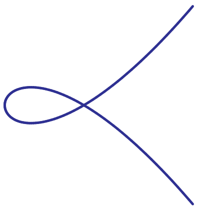
The most spectacular such bound concerns the number N_d of plane rational curves of degree d interpolating $3d-1$ points in $\mathbb{CP}^2$. $N_1 = 1$, as two points determine a line. Around 1990 Kontsevich gave the elegant recursion that starts with $N_1 = 1$,

$$N_d \;=\; \sum_{a+b=d} N_a N_b \left(a^2b^2\binom{3d-4}{3a-2} - a^3b\binom{3d-4}{3a-1}\right)\ .$$

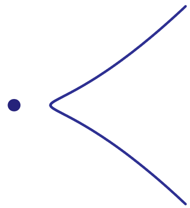
If the points lie in $\mathbb{RP}^2$, how many of the curves C can be real?

Welchsinger invariant

Real rational curves have 3 types of singularities:



node



solitary point

Welchsinger invariant

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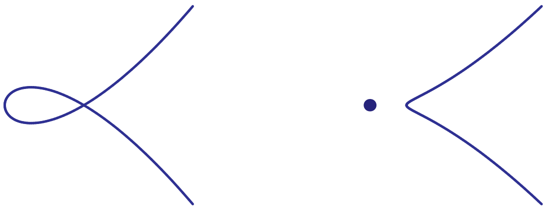


node　　　solitary point　　　complex conjugate nodes

Welschinger's Theorem. *The sum*

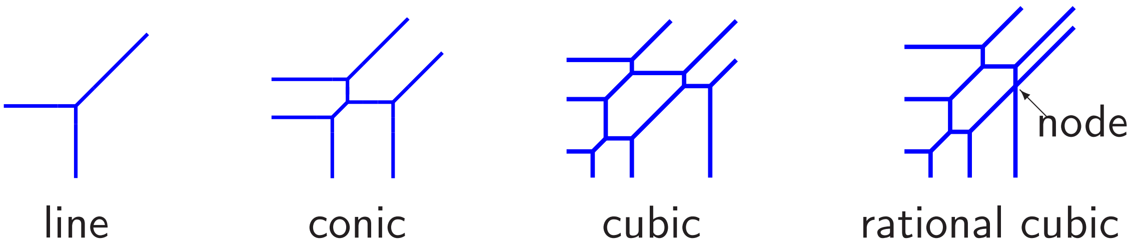
$$\sum (-1)^{\#\ \textit{solitary points in}\ C}\,,$$

over all real C interpolating $3d-1$ points in $\mathbb{RP}^2$, is independent of the choice of points.

This number is the Welschinger invariant, W_d.
$|W_d|$ is a lower bound for the number of interpolating real curves.

Tropical interpolation

Mikhalkin gave different formulae for N_d and W_d via counting tropical curves with multiplicities.



Using this, Itenberg, Kharlamov, and Shustin obtained the following:

- $W_d \geq \frac{d!}{3}\ (> 0)$,
- $\lim_{d\to\infty} \log(N_d)/\log(W_d) \ = \ 1$,
- A recursion for W_d.

Recently, Solomon showed W_d is the degree of a map.

Wronski map

The Wronskian, Wr $:= \det|(\frac{d}{dt})^i f_j(t)|$ of univariate polynomials $f_1(t), \ldots, f_k(t)$ of degree d, has degree $k(d+1-k)$. Up to a scalar, it depends only on the linear span of the f_j, and defines a map

$$\text{Wr} \;:\; \text{Grass}(k, d+1) \;\longrightarrow\; \mathbb{CP}^{k(d+1-k)}$$

of degree the number of Young tableaux of shape $k \times (d+1-k)$.

A Young tableau T is a linear extension of the product of chains of lengths k and $d+1-k$ and therefore has a sign $\sigma(T) \in \pm 1$.

Theorem (Eremenko-Gabrielov). If $W(x) \in \mathbb{RP}^{k(d+1-k)}$, then

$$\#\text{Wr}_{\mathbb{R}}^{-1}(W(x)) \;\geq\; |\sum_T \sigma(T)|\,.$$

Proof. The degree of real Wronski map equals RHS ($=$ sign-imbalance of product of chains of lengths k and $d+1-k$).

Lower bounds for sparse systems

Soprunova and I began to develop a theory of lower bounds for systems of polynomials with support $\mathcal{A}$. Its first step was to formulate a polynomial system as the fiber of a map $\pi\colon X_{\mathcal{A}} \to \mathbb{RP}^n$ from a real toric variety $X_{\mathcal{A}}$.

- $\longrightarrow$ Give conditions on $\mathrm{conv}(\mathcal{A})$ when π has a degree.
- $\longrightarrow$ Give a method to compute this degree in a special case (foldable triangulations).
- $\longrightarrow$ Use this method to compute degree for polynomial systems from a poset P, where the degree is the sign-imbalance of P.
- $\longrightarrow$ Use this and SAGBI degenerations to give new proof of Eremenko-Gabrielov theorem.

Joswig and Witte found many more examples coming from foldable triangulations.

When lower bound $=$ upper bound

The Wronski map

$$\mathrm{Wr} \ : \ \mathrm{Grass}(k, d{+}1) \ \longrightarrow \ \mathbb{CP}^{k(d+1-k)}$$

takes a k-plane of univariate polynomials to its Wronski determinant.

Theorem (Mukhin-Tarasov-Varchenko). *If a polynomial $\Phi \in \mathbb{RP}^{k(d+1-k)}$ has only real roots, then every k-plane in $\mathrm{Wr}^{-1}(\Phi)$ is real.*

Earlier, Eremenko and Gabrielov proved this when $k = 2$.

Theorem. *A rational function with real critical points is real.*

These results concern the Shapiro conjecture in Schubert calculus.

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Schubert calculus

A partition λ and a flag $F_\bullet$ in $\mathbb{C}^{d+1}$ together determine a Schubert variety, $X_\lambda F_\bullet \subset \mathrm{Grass}(k, d{+}1)$.

$|\lambda|$:= codimension of $X_\lambda F_\bullet$.

Given partitions $\lambda_1, \ldots, \lambda_m$ with $\sum |\lambda_i| = \dim(\mathrm{Grass})$ and general flags $F_\bullet^1, \ldots, F_\bullet^m$,

Kleiman's Theorem implies that

$$\bigcap_{i=1}^{m} X_{\lambda_i} F_\bullet^i$$

is transverse and consists of $d = d(\lambda_1, \ldots, \lambda_m)$ k-planes in $\mathbb{C}^{d+1}$.

Shapiro Conjecture

For $z \in \mathbb{C}$, the space $\mathbb{C}_d[t]$ of polynomials of degree $\leq d$ has a flag $F_\bullet(z)$ whose i-space consists of polynomials which vanish to order at least $d+1-i$ at z.

Shapiros's Conjecture (Mukhin-Tarasov-Varchenko Theorem).
If $z_1, \ldots, z_m$ are distinct and real, then

$$\bigcap_{i=1}^{m} X_{\lambda_i} F_\bullet(z_i)$$

is transverse and consists of $d(\lambda_1, \ldots, \lambda_m)$ real k-planes.

$\longrightarrow$ One proof (there are three) gives deep connection to representation theory.

$\rightsquigarrow$ Interesting combinatorial question concerning monodromy and Young tableaux.

Monodromy

The fibration

$$\bigcap_{i=1}^{m} X_{\lambda_i} F_\bullet(z_i) \ \longrightarrow \ (z_1, \ldots, z_m) \ \in \ (\mathbb{RP}^1) \setminus \Delta$$

has base with components necklaces (points z_i of $\mathbb{RP}^1$ labeled by partition λ_i) and each component is homeomorphic to $\mathbb{RP}^1$.

Fibers naturally labeled by an interesting set of Young tableaux.

Question. What is the monodromy?

Purbhoo has shown it is essentially Schützenberger evacuation and *jeu-de-taquin*.