Khovanskii-Rolle Continuation for Real Solutions

Computational Algebraic and Analytic Geometry

for Low-Dimensional Varieties

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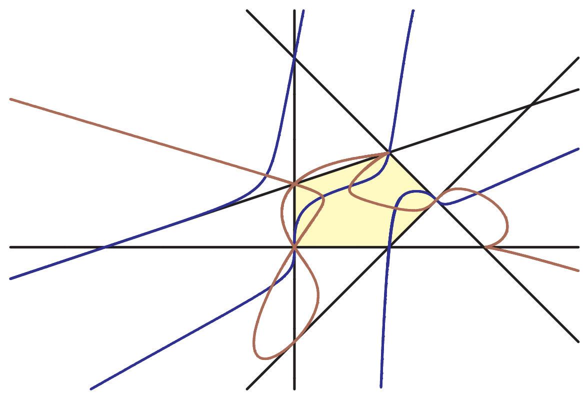


Our numerical future

Increasing parallelism $\Longrightarrow$ The future of computation in algebraic geometry is numerical.

We want to find all real solutions to a system of equations.

Current dominant numerical algorithm for solving, homotopy continuation, necessarily computes all solutions, both real and complex.

Two classes of numerical algorithms for real solutions:

— Exclusion methods.
Well-developed algorithms based on repeated subdivision.

— Semidefinite programming.
Recently proposed by Lasserre, Laurent, and Rostalski.

A third method

Khovanskii-Rolle continuation is a third numerical method to compute real solutions.

— Based on proof of fewnomial bounds for real solutions.

— Uses 2 symbolic steps:

1) Gale duality reduces a (potentially high-degree) polynomial system to a system of rational functions on a different space.

2) Reducing this to solving some systems of low-degree polynomials & some path-continuation.

— Complexity is essentially the fewnomial bound.

Gale duality, via example

Suppose we have the system of polynomials,

$$\begin{aligned} v^2w^3 &= 1 - u^2v - uv^2w\,, \\ v^2w &= \tfrac{1}{2} - u^2v + uv^2w\,, \\ uvw^3 &= \tfrac{10}{11}(1 + u^2v - 3uv^2w)\,. \end{aligned} \qquad (1)$$

Observe that

$$\begin{aligned} (u^2v)^2 \cdot (v^2w^3)^3 &= (uv^2w)^2 \cdot (v^2w) \cdot (uvw^3)^2 \quad \text{and} \\ (uv^2w)^3 \cdot (v^2w^3) &= (u^2v) \cdot (v^2w)^3 \cdot (uvw^3)\,. \end{aligned}$$

Substituting (1) into this, writing x for u^2v and y for uv^2w, and solving for 0, gives the Gale system of master functions

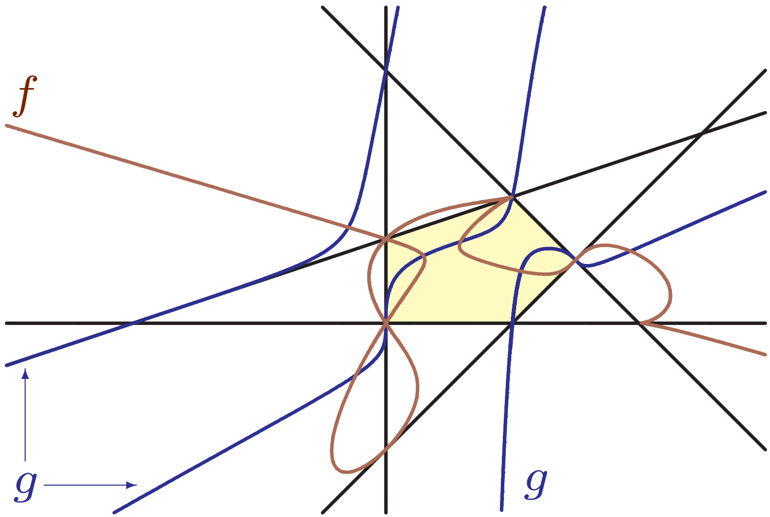
$$\begin{aligned} f &:= x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,, \\ g &:= y^3(1-x-y) \; - \; x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,. \end{aligned}$$

Gale duality, continued

The original system is equivalent to the Gale system

$$f := x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,,$$
$$g := y^3(1-x-y) \ - \ x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,,$$

in the complement of the lines given by the linear factors.



Khovanskii-Rolle continuation

Given a system of master functions

$$\prod_{i=1}^{\ell+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, \ldots, \ell\,, \tag{$*$}$$

($p_i(x)$ linear), we find solutions in the polyhedron

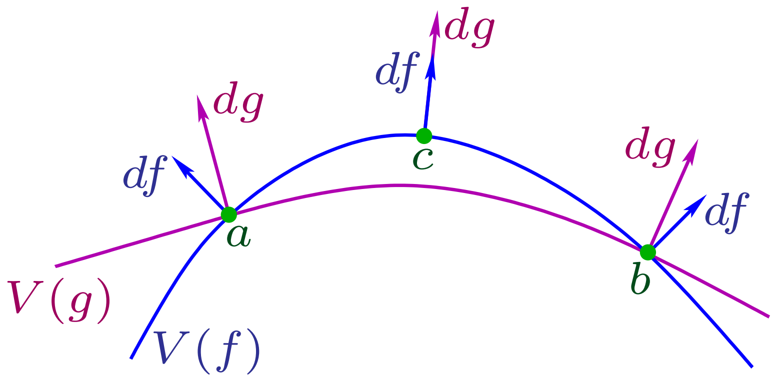
$$\Delta \;:=\; \{x \in \mathbb{R}^\ell \mid p_i(x) > 0\}\ .$$

The Khovanskii-Rolle Theorem (next slide) reduces solving $(*)$ to solving low degree polynomial systems, together with path continuation.

This is our new algorithm, which we now explain.

Khovanskii-Rolle Theorem

Theorem. *Between any two zeroes of g along the curve $V(f)$: $f=0$, lies at least one zero of the Jacobian $df \wedge dg$.*



Starting where $V(f)$ meets the boundary of the polyhedron Δ and where the Jacobian vanishes on $V(f)$, tracing the curve $V(f)$ in both directions finds all solutions $f = g = 0$.

Degree reduction ($\ell = 2$)

A system of master functions

$$\prod_{i=1}^{2+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, 2$$

in logarithmic form

$$\varphi_j \;:=\; \sum_{i=1}^{2+n} a_{i,j} \log p_i(x) \;=\; 0 \qquad j = 1, 2\,,$$

has Jacobians of low degree

$$J_2 \;:=\; \mathrm{Jac}(\varphi_1, \varphi_2) \qquad J_1 \;:=\; \mathrm{Jac}(\varphi_1, J_2)\ .$$

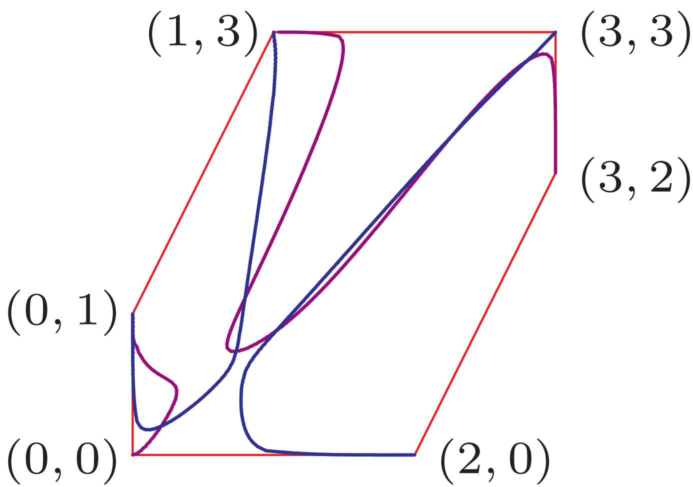
Here, $n = \deg(J_2)$ and $2n = \deg(J_1)$.

An example

Consider the system with $\ell = 2$ and $n = 4$:

$$f_1 \;:=\; \frac{(3500)^{12}x^{27}(3-x)^8(3-y)^4}{y^{15}(4-2x+y)^{60}(2x-y+1)^{60}} \;=\; 1\,,$$

$$f_2 \;:=\; \frac{(3500)^{12}x^{8}y^{4}(3-y)^{45}}{(3-x)^{33}(4-2x+y)^{60}(2x-y+1)^{60}} \;=\; 1\,.$$



Low-Degree Jacobians

If $\varphi_i := \log(f_i)$, then $J_2 := \mathrm{Jac}(\varphi_1, \varphi_2) \cdot \prod p_i(x,y) \; =$

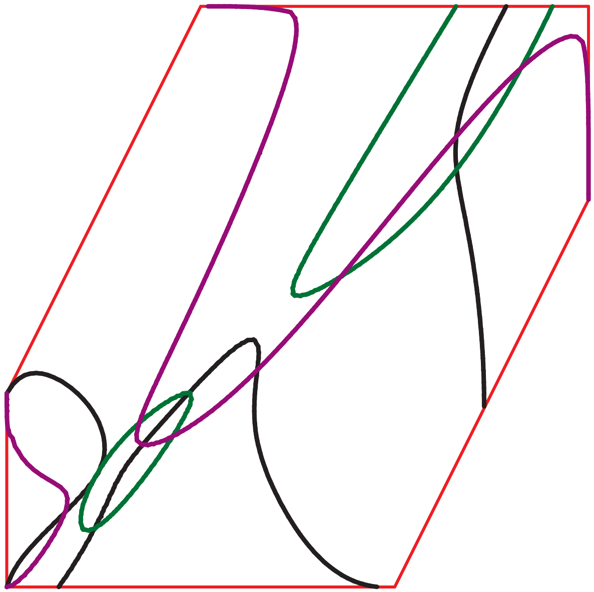
$$2736 - 15476x + 2564y + 32874x^2 - 21075xy + 6969y^2 - 10060x^3$$
$$-7576x^2y + 8041xy^2 - 869y^3 \; + \; 7680x^3y - 7680x^2y^2 + 1920xy^3\,.$$

(polynomial of degree $n = 4$.) $J_1 := \mathrm{Jac}(\varphi_1, \Gamma_2) \cdot \prod p_i(x,y)^2 \; =$

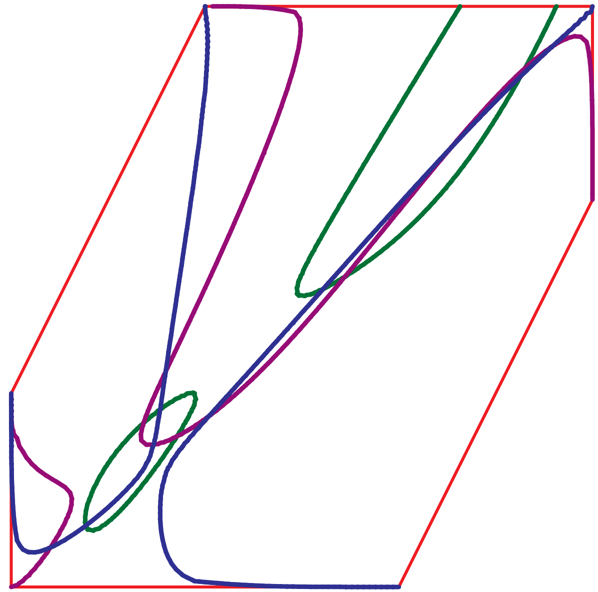
$$8357040x - 2492208y - 25754040x^2 + 4129596xy - 10847844y^2$$
$$-37659600x^3 + 164344612x^2y - 65490898xy^2 + 17210718y^3 \; + 75054960x^4$$
$$-249192492x^3y + 55060800x^2y^2 + 16767555xy^3 - 2952855y^4 \; - 36280440x^5$$
$$+143877620x^4y + 35420786x^3y^2 - 80032121x^2y^3 + 19035805xy^4 - 1128978y^5$$
$$+5432400x^6 - 33799848x^5y - 62600532x^4y^2 + 71422518x^3y^3 - 13347072x^2y^4$$
$$-1836633xy^5 + 211167y^6 + 2358480x^6y + 21170832x^5y^2 - 13447848x^4y^3$$
$$-8858976x^3y^4 + 7622421x^2y^5 - 1312365xy^6 - 1597440x^6y^2 - 1228800x^5y^3$$
$$+4239360x^4y^4 - 2519040x^3y^5 + 453120x^2y^6\,.$$

(A polynomial of degree $8 = 2n$.)

Completing the example



Follow $V(J_2) \cap \partial\Delta$ and $J_1 = J_2 = 0$ along $V(J_2)$ to find $J_2 = \varphi_1 = 0$.



Follow $V(\varphi_1) \cap \partial\Delta$ and $\varphi_1 = J_2 = 0$ along $V(\varphi_1)$ to find $\varphi_1 = \varphi_2 = 0$.