Galois Groups of Schubert Problems

Combinatorial Games and Schubert Calculus
MAA Mathfest, Pittsburgh
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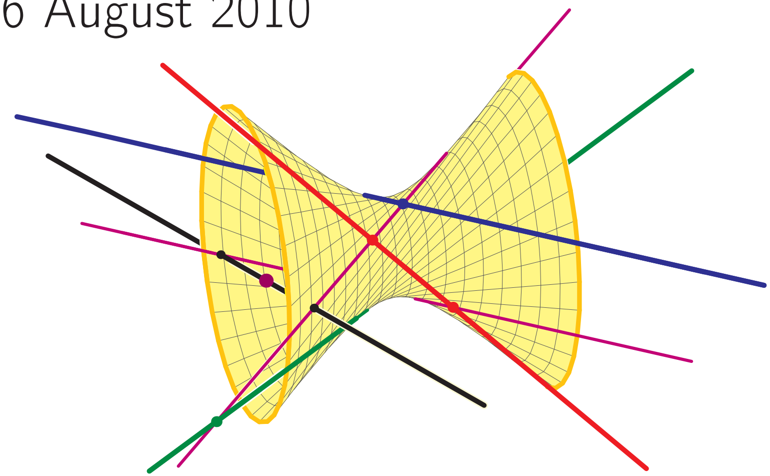

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Work with: Anton Leykin (Georgia Tech),
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The Problem of Four Lines

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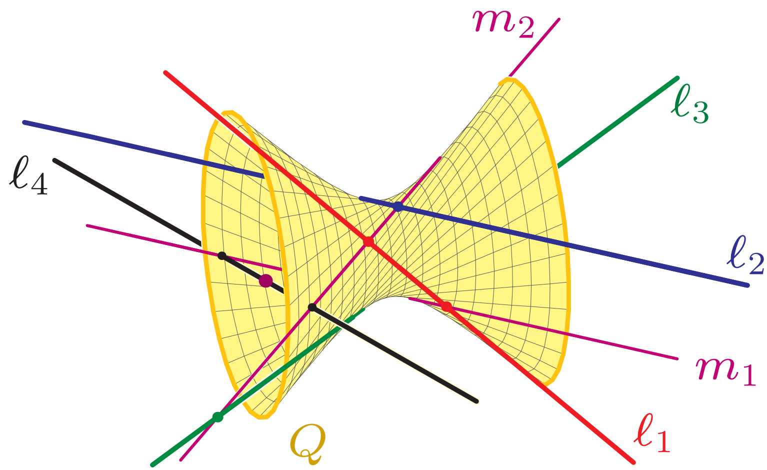
The lines ℓ_1, ℓ_2, ℓ_3 lie on a unique hyperboloid Q of one sheet, and the lines that meet ℓ_1, ℓ_2, ℓ_3 form one ruling of Q. Thus the solutions m_i are the lines in that ruling passing through the points of intersection $\ell_4 \cap Q$.



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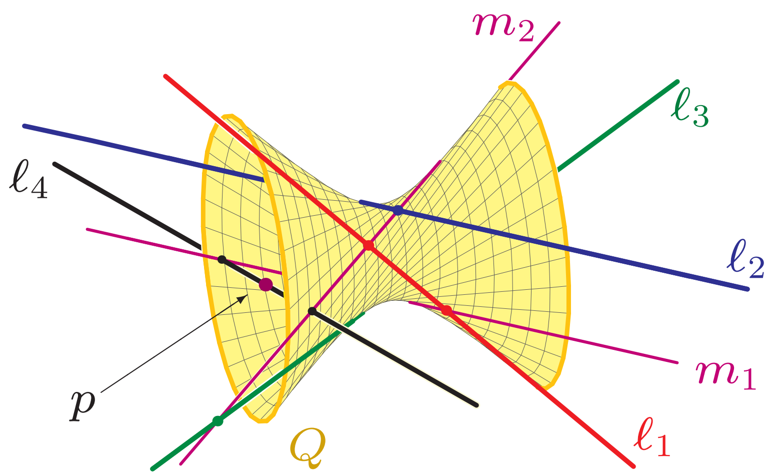

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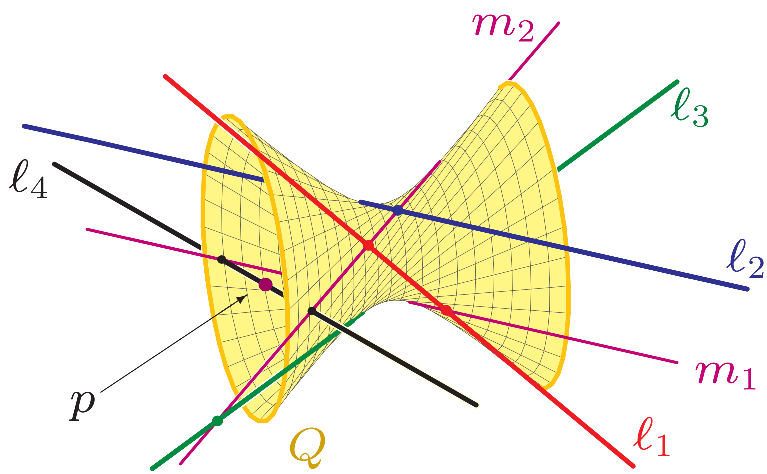


This shows that

The Galois group of the problem of four lines is the symmetric group $\mathcal{S}_2$.

A Problem with Exceptional Geometry

Q: What 4-planes H in $\mathbb{C}^8$ meet general 4-planes K_1, K_2, K_3, K_4?

Auxiliary problem: There are four (h_1, h_2, h_3, h_4) 2-planes in $\mathbb{C}^8$ meeting each of K_1, K_2, K_3, K_4. Schematically, $\square\square\square^{4} = 4$.

Fact: All solutions H to our problem have the form $H_{i,j} = \langle h_i, h_j \rangle$ for $1 \leq i < j \leq 4$. Schematically, $\boxplus^{4} = 6$.

It follows that the Galois group of $\boxplus^{4} = 6$ is equal to the Galois group of $\square\square\square^{4} = 4$, which is known to be the symmetric group $\mathcal{S}_4$.

This problem $\boxplus^{4} = 6$ also has exceptional reality: If K_1, K_2, K_3, K_4 are real, then either two or six of the $H_{i,j}$ are real, and never four or zero.

Galois Groups of Enumerative Problems

In 1870, Jordan explained how *algebraic* Galois groups arise naturally from problems in enumerative geometry, and in 1979 Harris showed that such an algebraic Galois group coincides with a geometric monodromy group.

This Galois group of a geometric problem is a subtle invariant. When it is *deficient* (i.e. not the full symmetric group), the geometric problem has some exceptional structure.

Work of Harris and of Vakil give several methods to determine Galois groups at least experimentally.

I will describe a project to study Galois groups using numerical algebraic geometry, symbolic computation, and combinatorics. We expect more questions than answers.

Some Theory

A degree e dominant map $E \xrightarrow{\pi} B$ of equidimensional irreducible varieties (up to codimension one, $E \to B$ is a covering space of degree e)
$\rightsquigarrow$ degree e field extension of fields of rational functions $K(E)/K(B)$, which has a Galois group $\mathrm{Gal}(E/B) \subset \mathcal{S}_e$.

Harris's Theorem. Restricting $E \to B$ to open subsets over which π is a covering space, $E' \to B'$, the Galois group is equal to the monodromy group of deck transformations.

This is the group of permutations of a fixed fiber induced by analytically continuing the fiber over loops in the base.

Point de départ: Such monodromy permutations are readily and reliably computed using methods of numerical algebraic geometry.

Enumerative Geometry

"Enumerative Geometry is the art of determining the number e of geometric figures x having specified positions with respect to other, fixed figures y."
— Hermann Cäser Hannibal Schubert, 1879.

B := configuration space of the fixed figures, and X := the space of the figures x we count. Then $E \subset X \times B$ consists of pairs (x, y) where $x \in X$ has given position with respect to $y \in Y$.

The projection $E \to B$ is a degree e cover outside of some discriminant locus, and the *Galois group of the enumerative problem* is $\mathrm{Gal}(E/B)$.

In the problem of four lines, B = four-tuples of lines, X = lines, and E consists of 5-tuples $(m, \ell_1, \ell_2, \ell_3, \ell_4)$ with m meeting each ℓ_i.

Proof-of-concept computation

Leykin and I used off-the-shelf numerical homotopy continuation software, and an implementation of (an easy version of) the Pieri homotopy algorithm to compute Galois groups of some Schubert problems formulated as the intersection of a skew Schubert variety with Schubert hypersurfaces.

In every case, we found enough monodromy permutations to generate the full symmetric group (determined by `Gap`). This included one Schubert problem with $e = 17,589$ solutions.

We conjecture that problems of this type will always have the full symmetric group as Galois group.

To investigate this question for more general Schubert problems, both on Grassmannians and on other flag manifolds, we need more algorithms and implementations.

Numerical Project

Recent work, including certified continuation (Beltrán and Leykin), Littlewood-Richardson homotopies (Vakil, Verschelde, and S.), regeneration (Hauenstein), implementation of Pieri and of Littlewood-Richardson homotopies (Martin del Campo and Leykin) will enable the reliable numerical computation of Galois groups of more general Schubert problems.

We plan to use a supercomputer whose day job is calculus instruction to investigate many of the millions of accessible and computable Schubert problems. Our intention is to build a library of Schubert problems (expected to be very few) whose Galois groups are deficient.

These data would be used to generate conjectures, leading to proofs about Galois groups of Schubert problems, as well as showcase the possibilities of numerical computation.

Vakil's Alternating Lemma

Suppose $S \subset B$ has a dense set of regular values of $E \to B$. Then

$$\mathrm{Gal}(E|_S/S) \hookrightarrow \mathrm{Gal}(E/B)\,.$$

This occurs often in enumerative geometry. Geometric degenerations

$$X \cap Y \rightsquigarrow W \cup Z$$

give natural families $S \subset B$ with

$$E|_S \simeq F \coprod_S G \quad \text{where } F \to S \text{ and } G \to S$$

are the sub-enumerative problems for W & Z of degrees f, g, where $f + g = e$.

Vakil's Alternating Criteria. If $f \neq g$ and both $\mathrm{Gal}(F/S)$ and $\mathrm{Gal}(G/S)$ contain the alernating groups A_f and A_g, then $\mathrm{Gal}(E/B)$ contains the alternating group A_e.

Application of Vakil's Criterion

Vakil's geometric Littlewood-Richardson rule and the earlier Pieri degenerations allow the use of Vakil's criterion.

Christopher Brooks has an efficient `python` script implementing Vakil's geometric Littlewood-Richardson rule and criterion. We intend to use it in our study of Galois groups. Managing mountains of data is a challenge.

For example, all Schubert problems in $G(2,n)$ with $n \leq 40$ have at least alternating Galois groups.

We hope to prove this for all $G(2,n)$ using a formula of Scherbak-Varchenko: $\sigma_{a_1} \cdot \sigma_{a_2} \cdots \sigma_{a_n} \cdot \sigma_{a_{n+1}}$ on $G(2,d)$ is

$$\sum_{I \subset [n]} (-1)^{n-|I|} \binom{a_I + |I| - d}{n-2} ,$$

where $a_I = \sum_{i \in I} a_i$.

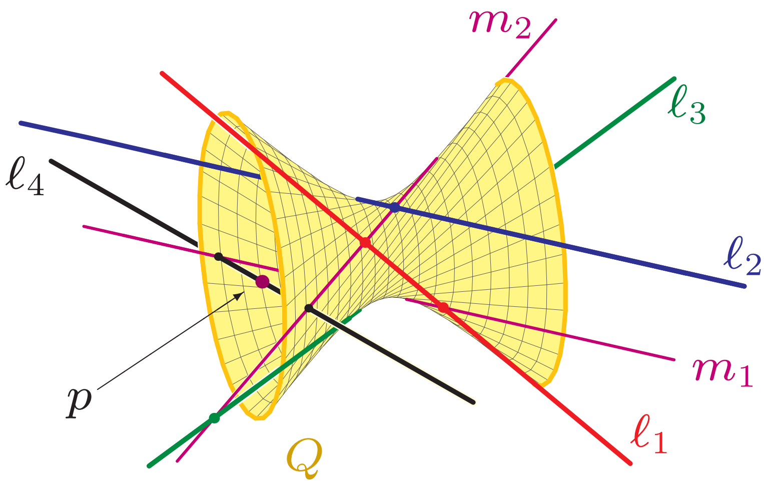
Specialization Lemma

Suppose that $\pi : E \to B$ with B rational. Then the fibre $\pi^{-1}(y)$ above a $\mathbb{Q}$-rational point $y \in B(\mathbb{Q})$ is has a minimal polynomial $p_y(t) \in \mathbb{Q}[t]$. In this situation, the algebraic Galois group of $p_y(t)$ is a subgroup of Gal(E/B).

This can be applied to Schubert problems (and many other geometric problems). The minimal polynomial of such fibers are easy to compute in many cases when $e \lesssim 50$, which should enable this method to be used to study Galois groups of many thousands to millions of Schubert problem.

The bottleneck is that software to compute Galois groups of the huge polynomials we generate cannot handle even moderate-sized polynomials.

Thank You!



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