Discriminant Coamoebas in Dimension 2 via Homology

Special Session on A-Hypergeometric functions

AMS Fall Western Section Meeting, October 23 2011

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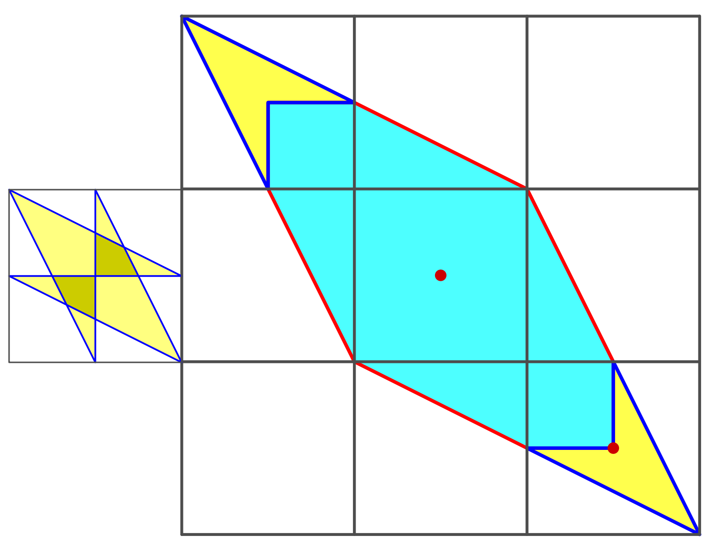

Mikael Passare 1.1.1959–15.9.2011

Michael Passare was a professor of mathematics at Stockholm University since 1992. He worked in several complex variables, turning in the past decade to the study of amoeboae and most recently to coamoebae.

Passare was the deputy director of the Institut Mittag-Leffler, chairman of the Swedish Mathematical Society, and former head (2005-2010) of the Department of Mathematics at Stockholm University.

Passare's advisor was Kiselman. Passare supervised 9 Ph.D. students, and had three current students. According to MathSciNet, Passare has 33 published papers with 14 co-authors.

Mikael Passare 1.1.1959–15.9.2011

He was noted for his warmth and friendship, and his contacts around the world. He led many of us at the Institut Mittag-Leffler to enjoy the Swedish out-of-doors.

Passare was a polyglot and linguist, suffused with a love of language. In our last meeting in August, he was thrilled that my description of how I was stuck led him to a new word, stymied.

He adopted his surname, Passare, which means compass in Swedish, as an adult. This was prescient, for his most recent work concerns the angles in complex varieties. This is the topic of my talk, which grew out of our last meeting 6 August 2011.

Connection to A-Hypergeometric Functions

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A-Discriminants

Let $A \subset \mathbb{Z}^n$ have $N+1 := n+d+1$ elements. The variety in $(\mathbb{C}^*)^n$ of a polynomial with support A

$$f \;:=\; \sum_{\mathbf{a}\in A} c_{\mathbf{a}} z^{\mathbf{a}}$$

is smooth when the coefficients do not lie in a hypersurface $\Sigma_A \subset \mathbb{R}^A$, called the *$A$-discriminant*.

The A-discriminant has many homogeneities. Quotienting by these homogeneities gives the *reduced A-discriminant Σ_A^r*, which is a hypersurface in $\mathbb{C}^d$.

Our goal is to understand the phases in the reduced A-discriminant.

Horn-Kapranov Parametrization

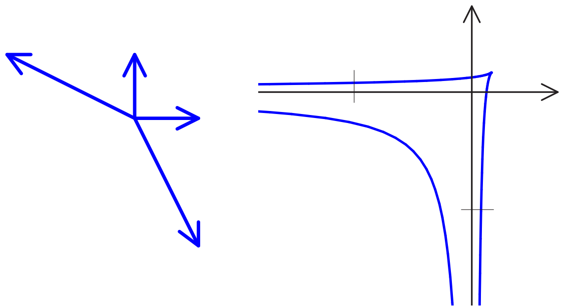
Let $B = \{b_0, b_1, \ldots, b_N\} \subset \mathbb{Z}^d$ be a homogeneous Gale transform of A. That is, if $\mathcal{B}$ is the matrix whose rows are the vectors in B, the columns of $\mathcal{B}$ are a basis for the $\mathbb{Z}$-module of homogeneous linear relations among A—those whose coefficents have sum 0.

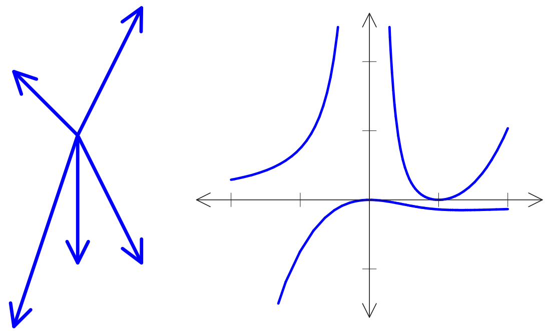
Then the reduced A-discriminant $\Sigma_A^r \subset \mathbb{C}^d$ is the image of the rational map $\Psi\colon \mathbb{P}^{d-1} \longrightarrow \mathbb{C}^d$ defined by

$$\Psi([t_1 : \cdots : t_d]) \;=\; \left(\prod_{i=0}^{N} \langle t, b_i \rangle^{b_{i,1}}, \ldots, \prod_{i=0}^{N} \langle t, b_i \rangle^{b_{i,d}}\right)$$

Some A-discriminants

Here are some reduced A-dicriminants in dimension $d = 2$, together with a Gale dual vector configuration:





Amoebae and coamoebae

The nonzero complex numbers $\mathbb{C}^*$ decompose

$$\mathbb{R} \times \mathbb{T} \xrightarrow{\sim} \mathbb{C}^* \qquad (r, \theta) \mapsto e^{r+\sqrt{-1}\theta},$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the unit complex numbers.

The inverse map $z \mapsto (\log|z|, \arg(z))$ induces maps

$$\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n \quad \text{and} \quad \text{Arg} : (\mathbb{C}^*)^n \to \mathbb{T}^n.$$

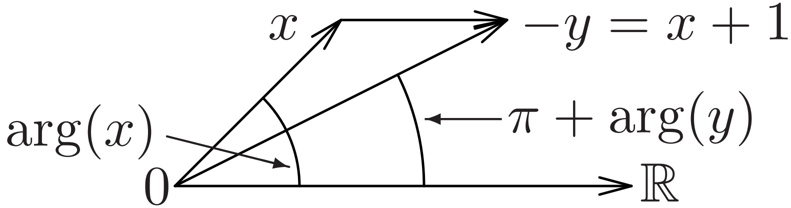
Amoebae and *coamoebae* are the images of algebraic subvarieties of the torus $(\mathbb{C}^*)^n$ under the maps Log and Arg, respectively.

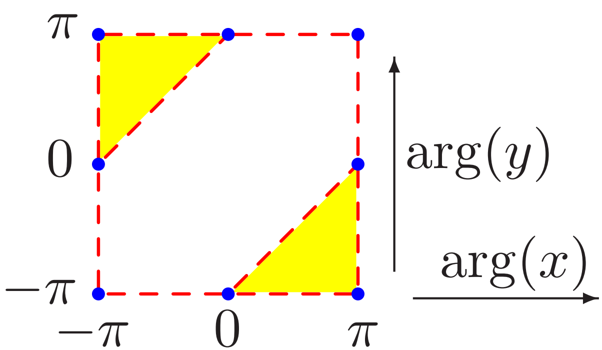
This talk is concerned with coamoebae of reduced A-discriminants, when $d = 2$.

Coamoeba of ℓ: $x + y + 1 = 0$

Points of the coamoeba of ℓ are $(\arg(x), \pi + \arg(x+1))$. Real points map to $(\pi, 0)$, $(0, \pi)$, and (π, π).

For other points, consider the picture in the complex plane





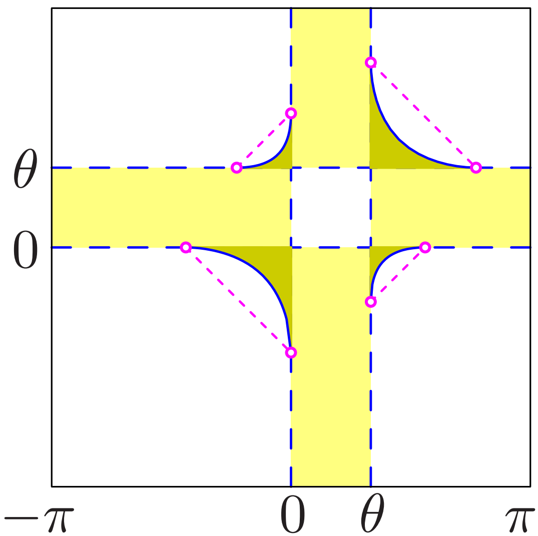
Thus $\pi + \arg(y)$ lies between 0 and $\arg(x)$, and the coamoeba of ℓ consists of the three real points and the interiors of the two triangles.

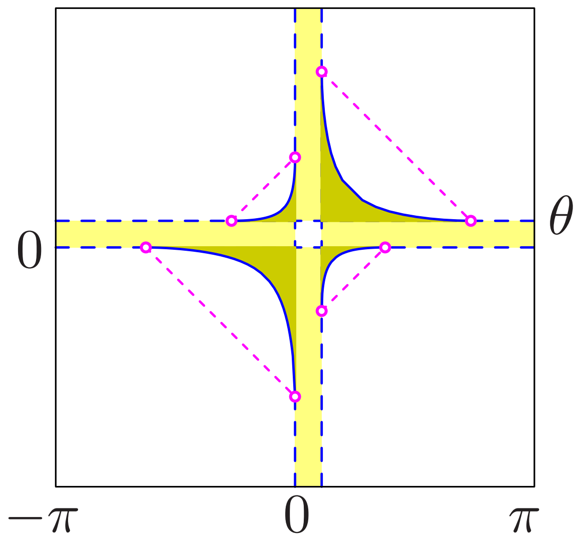
More Coamoebae

Coamoebae are not necessarily composed of polyhedra. We show coamoebae of two hyperbolae

$$a \;-\; x \;-\; y \;+\; xy \;=\; 0\,,$$

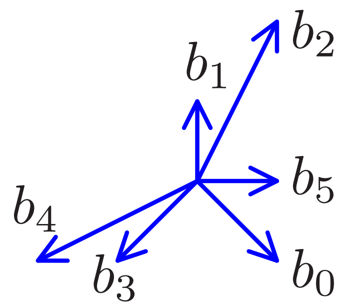
for different values of $a = \rho e^{\sqrt{-1}\theta}$.





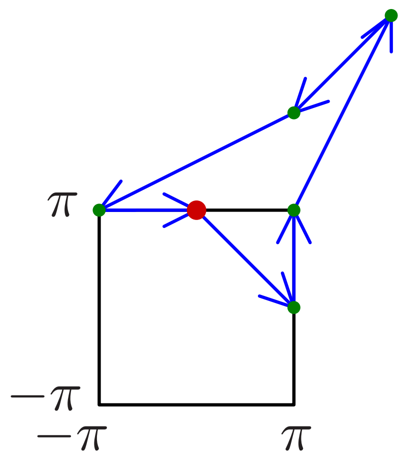
Nilsson-Passare description of A-discriminant coamoebae

Order the vectors of B by the clockwise order of the lines they span, starting from just below the horizontal.



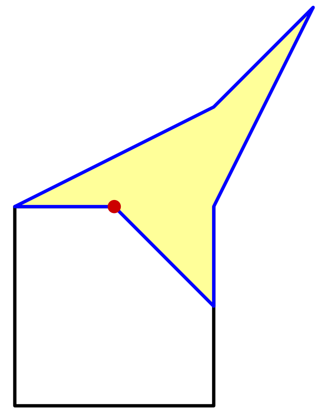
We describe the A-discriminant coamoeba in $\mathbb{R}^2$, the universal cover of $\mathbb{T}^2$.

Starting at which of $(0, 0)$, $(\pi, 0)$, $\boldsymbol{(0, \pi)}$, (π, π) is the argument of the Horn-Kapranov parametrization at $[1, t]$ for $t \gg 0$, place the vectors $\pi b_0, \ldots, \pi b_N$ in order, head-to-tail.



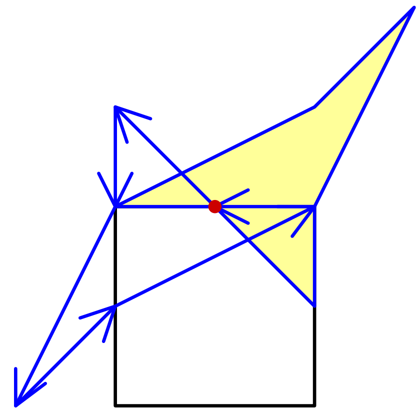
A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane.



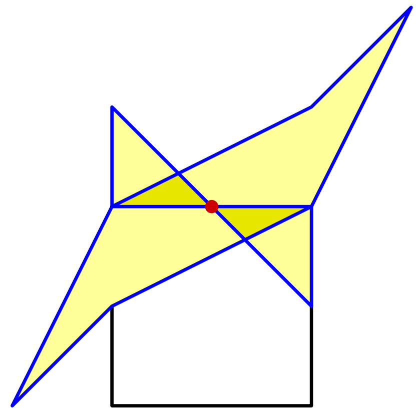
A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order.



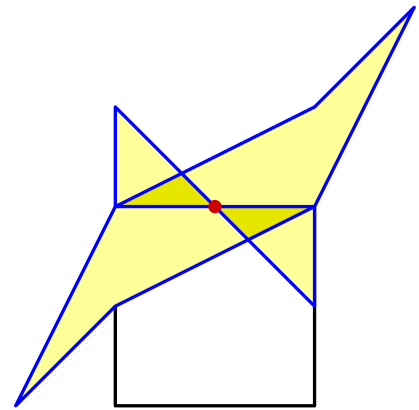
A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order, and fill in to get the rest of the *coamoeba chain* $\mathcal{A}_B$.



$\mathcal{A}$-discriminant coamoebae

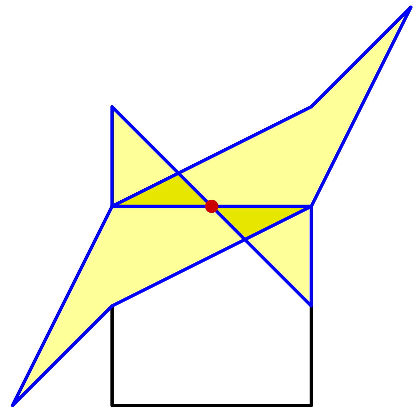
This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order, and fill in to get the rest of the *coamoeba chain* $\mathcal{A}_B$.



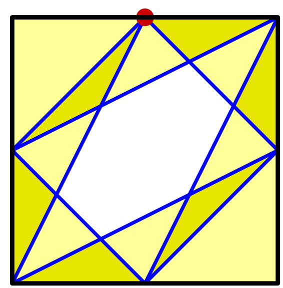
This is an oriented topological 2-chain in $\mathbb{T}^2$ whose oriented boundary consists of the edges coming from $\pi b_0, \ldots, \pi b_N, -\pi b_0, \ldots, -\pi b_n$.

$\mathcal{A}$-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order, and fill in to get the rest of the *coamoeba chain* $\mathcal{A}_B$.

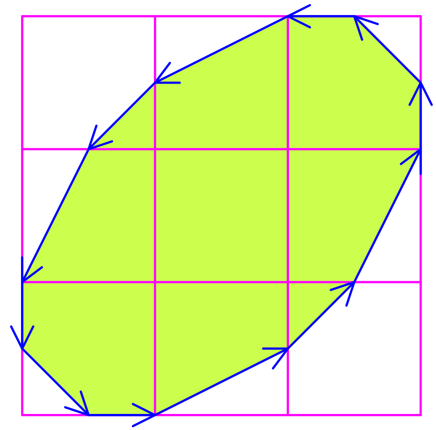


This is an oriented topological 2-chain in $\mathbb{T}^2$ whose oriented boundary consists of the edges coming from $\pi b_0, \ldots, \pi b_N, -\pi b_0, \ldots, -\pi b_n$.
We display it in the fundamental domain.



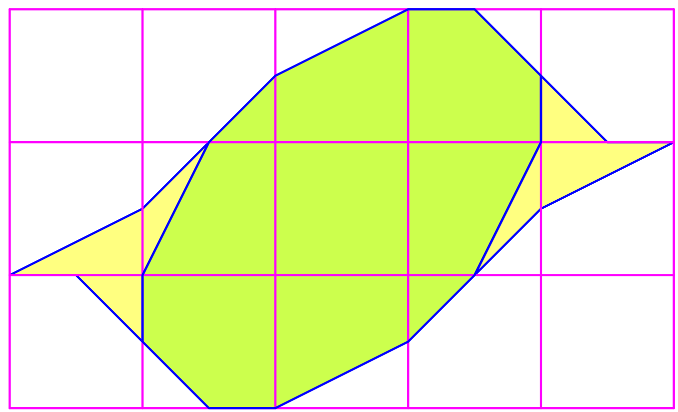
A zonotope

The boundary of this coamoeba chain, but in reverse order, is shared by the zonotope Z_B generated by the vectors $\pi b_0, \pi b_1, \ldots, \pi b_N$.



The union of Z_B and the coamoeba chain is therefore a topological 2-cycle on $\mathbb{T}^2$.

In our example, $\mathcal{A}_B \cup Z_B$ covers 7 fundamental domains.



Nilsson-Passare Theorem

The main results of Nilsson-Passare included this description of the coamoeba—perhaps surprisingly, it is an object from geometric combinatorics.

From this they deduced a formula for the area of the coamoeba chain $\mathcal{A}_B$.

The area of the zonotope Z_B has a known formula.

Their last result is that the sum of the area of $\mathcal{A}_B$ and of Z_B is equal to the $(2\pi)^2$ times the normalized volume of the convex hull of the original point configuration A.

The proof of the last result was unsatisfactory, for it did not explain the result, and it absolutely does not generalize to $d > 2$.

Factorization of the Horn-Kapranov parametrization

Mikael and I looked for a second proof based on a factorization of the Horn-Kapranov parametrization, $\Psi\colon \mathbb{P}^1 \to \mathbb{C}^2$,

$$\Psi([s:t]) \;=\; \left(\prod_{i=0}^{N} \langle t, b_i\rangle^{b_{i,1}}, \prod_{i=0}^{N} \langle t, b_i\rangle^{b_{i,2}}\right)\;.$$

$$\begin{array}{ccccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^N & \dashrightarrow & (\mathbb{C}^*)^2 \\ t & \longmapsto & \big(\langle t, b_0\rangle, \ldots, \langle t, b_N\rangle\big) & & \\ & & [x_0, \ldots, x_N] & \longmapsto & \left(\prod_i x_i^{b_{i,1}}, \prod_i x_i^{b_{i,2}}\right) \end{array}$$

The first map is a parametrization of a line, and the second is a homomorphism of dense tori. We examine the effect of each map on coamoebae.

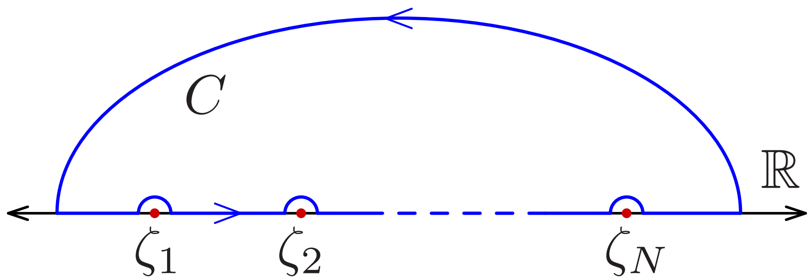
Coamoeba of a real line

For $B \subset \mathbb{Z}^2$, we describe the coamoeba of the image of

$$\psi : \mathbb{P}^1 \ni t \longmapsto \left(\langle t, b_0\rangle, \ldots, \langle t, b_N\rangle\right).$$

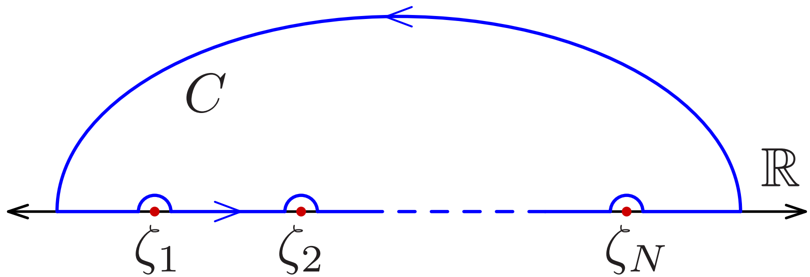
Assume that the zeroes ζ_i of the linear functions $\langle \bullet, b_i\rangle$ are in order with $\zeta_0 = \infty$ and $\zeta_1 < \zeta_2 < \cdots < \zeta_N$.

Consider $\mathrm{Arg} \circ \psi$ on a countour C in the upper half plane



This is constant on intervals of $\mathbb{R}$, on each small arc it increments by $-\pi\mathbf{e}_i$, and by $\pi\mathbf{e}_1 + \cdots + \pi\mathbf{e}_N$ on the arc near ∞.

Coamoeba of a real line



$\mathrm{Arg} \circ \psi$ is constant on intervals of $\mathbb{R}$ and on each small arc increments by $-\pi\mathbf{e}_i$ and $\pi\mathbf{e}_1 + \cdots + \pi\mathbf{e}_N$ on the arc near ∞.

This describes a piecewise linear closed path in $\mathbb{T}^N$ starting at $\mathrm{Arg}(\psi(t))$ for $t \ll 0$ and enclosing the coamoeba of the image of the upper half plane. The coamoeba of the lower half plane is obtained by multiplication by -1.

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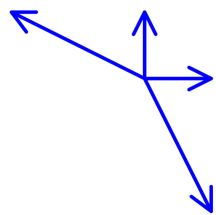

Zonotope chain

The closure of the coamoeba of the line $\mathcal{A}_\ell := \mathrm{Arg}\circ\psi(\mathbb{P}^1)$ is a chain in $\mathbb{T}^N$. There is a second chain with the same boundary as $\overline{\mathcal{A}_\ell}$ but with opposite orientation.

The *zonotope chain* Z_ℓ is the union of triangles (or segments) formed by the convex hull of $0 \in \mathbb{T}^N$ and each of the segments bounding $\overline{\mathcal{A}_\ell}$.

The union $\overline{\mathcal{A}_\ell} \cup Z_\ell$ is a topological 2-cycle in $\mathbb{T}^N$.

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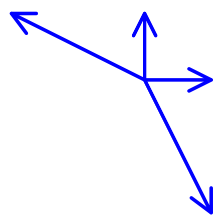

A-discriminant coamoeba

Write φ for the homomorphism $\mathbb{T}^N \to \mathbb{T}^2$ induced by

$$[x_0, \ldots, x_N] \longmapsto \Big(\prod_i x_i^{b_{i,1}}, \prod_i x_i^{b_{i,2}}\Big)\,.$$

Note that φ maps the coordinate vector $\pi\mathbf{e}_i$ to πb_i.

Then the coamoeba chain $\mathcal{A}_B$ of the reduced A-discriminant is simply the push-forward $\varphi_*(\overline{\mathcal{A}_\ell})$.

Similarly, the zonotope Z_B in $\mathbb{T}^2$ is the pushforward $\varphi_*(Z_\ell)$ of the zonotope chain.

Then $\varphi_*(\overline{\mathcal{A}_\ell} \cup Z_\ell) = k[\mathbb{T}^2]$, where k measures how often this pushforward covers $\mathbb{T}^2$.

Enter homology

We compute $\varphi_*(\overline{\mathcal{A}_\ell} \cup Z_\ell) = k[\mathbb{T}^2]$ by determining the homology class of $\overline{\mathcal{A}_\ell} \cup Z_\ell$. For this, we determine its pushforward to each 2-dimensional coordinate projection.

This gives a formula $[\overline{\mathcal{A}_\ell} \cup Z_\ell] = \sum \mathbf{e}_i \wedge \mathbf{e}_j$, where the sum is over certain $i < j$.

Since $\varphi_*(\mathbf{e}_i) = b_j$ on H_1, we obtain a formula of the form

$$\varphi_*[\overline{\mathcal{A}_\ell} \cup Z_\ell] \;=\; \sum (b_i \wedge b_j)[\mathbb{T}^2]\,.$$

The precise form of this sum is an expression for the normalized volume of the convex hull of the point configuration A Gale dual to B, giving a new proof of the Nilsson-Passare Theorem.

Bibliography

L. Nilsson and M. Passare, "Discriminant coamoebas in dimension two", `arXiv:0911.0475`.

M. Passare and F. Sottile, "Discriminant coamoebas in dimension two through homology", in progress.