

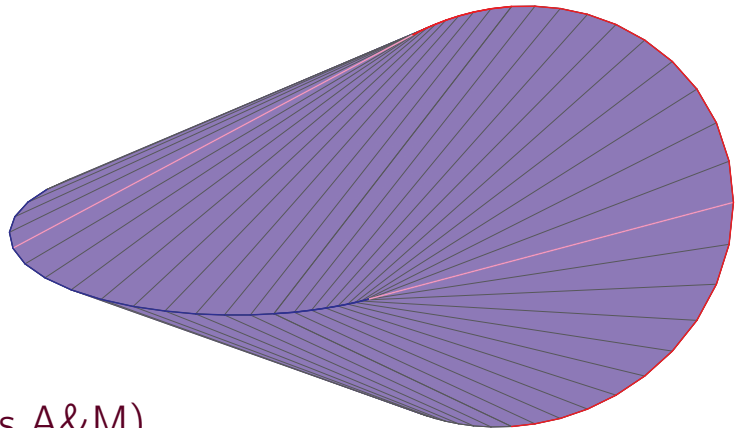
# The Convex Hull of a Pair of Circles

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Work with Tina Mai (Texas A&M),

# Convex Algebraic Geometry

Convex algebraic geometry studies convex hulls of semi-algebraic sets from the perspectives of

- Optimization
- Classical convexity
- Algebraic geometry

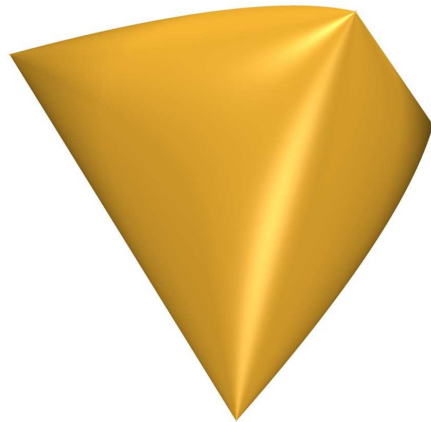
For example, the object on the right is the bounded region of Cayley's cubic,

$$x^2 + y^2 - x^2z + y^2z + z^2 - 1,$$

dubbed the [samosa](#) by Raman Sanyal.

It is a spectrahedron, its faces are the four singular vertices, the six line segments joining them, and the remaining points of its boundary. All are exposed.

Our goal is to describe the boundary of certain convex hulls, as real algebraic varieties.



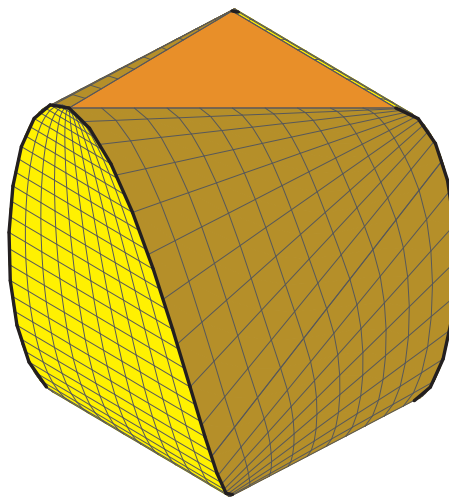
# Convex Hulls of Space Curves

Ranestad and Sturmfels<sup>‡</sup> showed that the boundary of a space curve consists of *tritangent planes* and a family of *stationary bisecants*—bisecants joining points whose tangent lines meet.

Here is the convex hull of the trigonometric curve,

$$(\cos(\theta), \sin(2\theta), \cos(3\theta)) ,$$

which is cut out by  $4x^2 - 3x - z$  and  $y^2 - 4x^2 + 4x^4$ . It has two tritangent planes (giving triangular faces), and two algebraic families of stationary bisecants, whose rulings are shown.



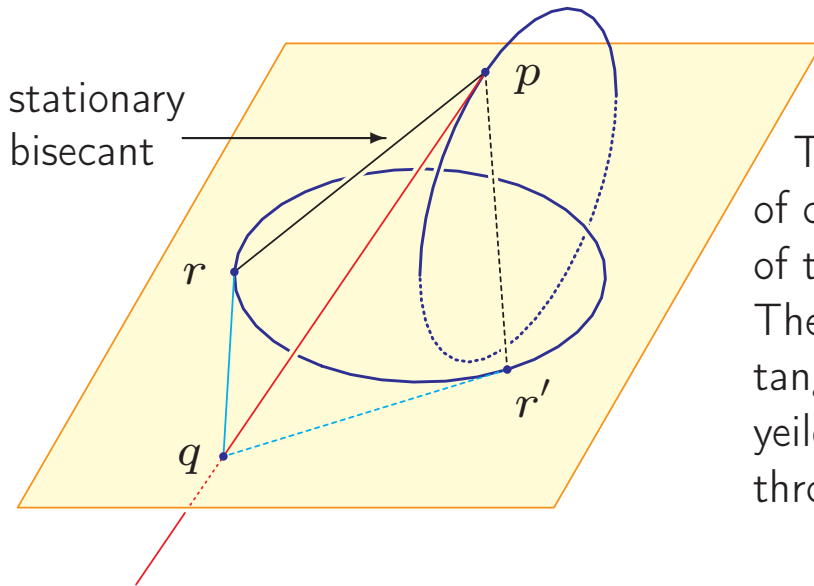
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<sup>‡</sup> *On the convex hull of a space curve*, Adv. Geom., 12 (2009), 157–178.

# Stationary Bisecants to Two Circles

We consider the convex hull of two circles in  $\mathbb{R}^3$ . The discs of the circles may or may not lie in the boundary of the convex hull.

A general point  $p$  of one circle lies on two stationary bisecants:



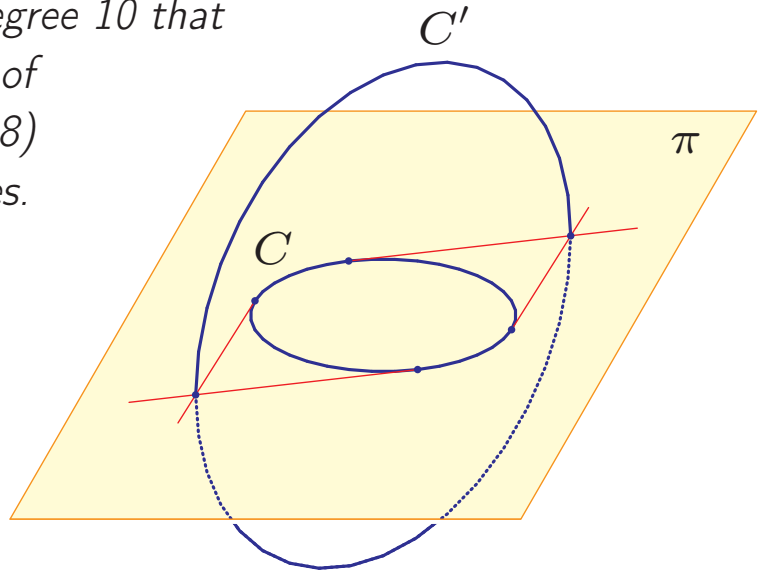
The tangent to a point  $p$  of one circle meets the plane of the second in a point  $r$ . The second circle has two tangents through  $r$ , each yielding a stationary bisecant through  $p$ .

# Boundary of Convex Hull of Two Circles

**Theorem.** *The algebraic closure of the boundary of the convex hull of two circles is reducible surface of degree 10 that consists of the ruled surface  $\mathcal{S}$  of stationary bisecants (of degree 8) and the two planes of the circles.*

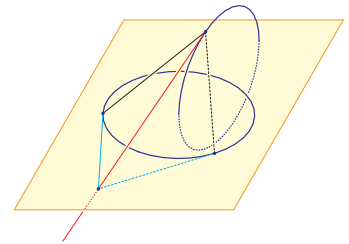
**Proof.**  $\mathcal{S}$  is singular along each conic. We compute the degree of  $\mathcal{S}$  by intersecting it with the plane  $\pi$  of one conic  $C$ . This gives  $2C$  and four tangents to  $C$  meeting  $C' \cap \pi$ .

This implies that  $\mathcal{S}$  has degree  $2 \cdot 2 + 4 \cdot 1 = 8$ .



# Edge Curve of Stationary Bisecants

In the product of circles ( $\simeq \mathbb{R}P^1 \times \mathbb{R}P^1$ ), the endpoints  $(p, q)$  of stationary bisecants form the *edge curve*  $\mathcal{E}$ .



$\mathcal{E}$  has a 2-to-1 projection to each factor  $\Rightarrow \mathcal{E}$  is a curve of bidegree  $(2, 2)$  and therefore has arithmetic genus one.

Q: What possibilities for  $\mathcal{E}$  occur?

If  $\mathcal{E}$  is smooth, it is an irreducible genus one curve.

If  $\mathcal{E}$  is singular and irreducible, it is either a nodal or a cuspidal rational curve.

What reducible types of edge curves  $\mathcal{E}$  can occur?

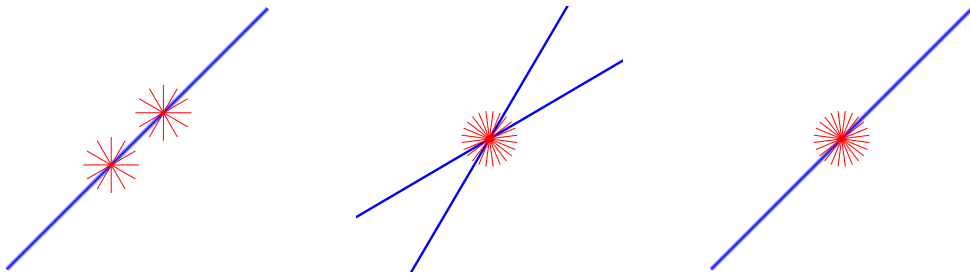
# Some pictures

.html Gallery

# Beginnings of Classification of $\mathcal{E}$

We do know something about these classification questions.

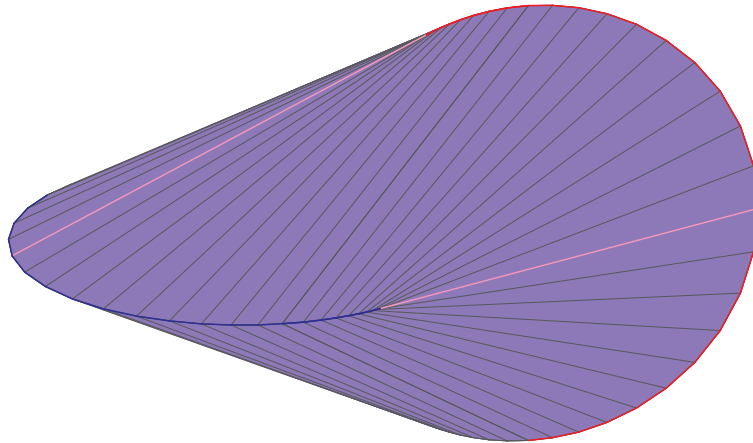
- All  $j$ -invariants of genus one curves occur. (likely all real forms, too).
  - At most nodal singularities. (cusps are impossible **Surprising!**).
  - Many reducible types occur, and we know what they are and why.
- Likely all noncuspidal  $(2, 2)$ -curves occur if we look at stationary bicants to pairs of (planar) complete conics:





# More to do

- Complete the classification.
- Compute the volume of the convex body.
- Area of the edge surface.
- Combinatorial classification of boundaries. This should also extend to convex hulls of pairs of ovoids.
- Write about a dynamical system on a pair of circles and Poncelet's Theorem.



# Thank You!

