Fermi Isospectrality

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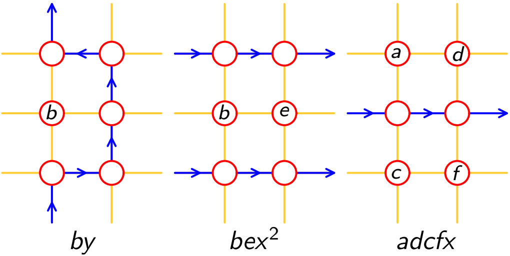

Work with Matthew Faust and Wencai Liu

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Floquet and Fermi Varieties

A *Schrödinger operator* $H = \Delta + P$ on a graph $\Gamma = (V, E)$ acts on $\mathbb{R}^V := \{f\colon V \to \mathbb{R}\}$ with Δ a second order difference operator and P a potential. For $f \in \mathbb{R}^V$ and $v \in V$, we have

$$(Hf)(v) = \sum_{(u,v)\in E} c_{u,v}(f(v) - f(u)) \ + \ P(v)f(v)\,.$$

Suppose $\mathbb{Z}^d$ acts on Γ (and P, $\{c_{u,v}\}$) with finitely many orbits.

For a character $z\colon \mathbb{Z}^d \to S^1$ ($z \in (S^1)^d$) let

$$\ell_z(\Gamma) := \{f \in \mathbb{R}^V \mid f(v+\alpha) = z^\alpha f(v), v \in V, \alpha \in \mathbb{Z}^d\}\,,$$

which is a finite-dimensional vector space.

Then H restricts to H_z on $\ell_z(\Gamma)$.

The *Floquet variety* is $\{(z,\lambda) \mid \exists f \in \ell_z(\Gamma)\ H_z f = \lambda f\}$.

A *Fermi variety* is a level set (λ fixed) of the Floquet variety.

Inverse Problem

To what extent do the Floquet/Fermi varieties determine the parameters (potential P and edge weights $\{c_{u,v}\}$) ?

This question is algebraic, so we extend coefficients to $\mathbb{C}$ ($f \in \mathbb{C}^V$).

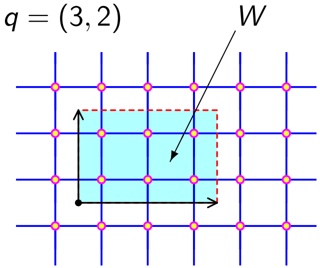
In appropriate coordinates, $(z, \lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C}$, the Floquet and Fermi varieties are algebraic hypersurfaces given by the vanishing of a single polynomial $F(z, \lambda)$.

A common test case for these questions is the grid graph Γ, which is $\mathbb{Z}^d$ with a *box-periodic potential:*

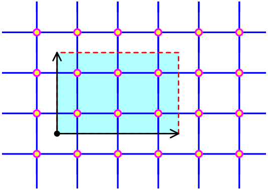
$$P(u + q_i e_i) = P(u)$$

where $q_1, \ldots, q_d \in \mathbb{N}$ and e_i is a standard basis vector, (and the same for $\{c_{u,v}\}$).

Let W be a fundamental domain for the action of $q_1\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$.



Box-Periodic Floquet Isospectrality



Let Γ be the grid graph $\mathbb{Z}^d$ and let the group acting on it be $q_1\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z} \subset \mathbb{Z}^d$.

Kappeler, Duke & Adv. Appl. Math. (1988): Suppose that $d \geq 2$ and Δ is fixed.

1. Knowing $F_P(1, \lambda)$ ($\leq N = q_1 \cdots q_d$ spectral values) determines the potential P up to permutations. There are at most $N!$ potentials Q with $F_Q(1, \lambda) = F_P(1, \lambda)$.
2. If Δ is the graph Laplacian, $c_{u,v} = 1$, and $P \in \mathbb{C}^W$ is generic, then $F_P(z, \lambda)$ determines P. For $z_1 \neq z_2$ in $(\mathbb{C}^\times)^d$, $F_P(z_1, \lambda)$ and $F_P(z_2, \lambda)$ together determine P.
3. When $d = 2$, if Δ a generic difference operator, then $F_P(z, \lambda)$ determines P, always.

Fermi Isospectrality

W. Liu, *Fermi Isospectrality . . .*, `arXiv:2106:03726`.

Definition. Two potentials $P, Q \in \mathbb{C}^W$ are *Fermi isospectral* if there is a $\lambda_0 \in \mathbb{C}$ such that

$$F_P(z, \lambda_0) \ = \ F_Q(z, \lambda_0)\,.$$

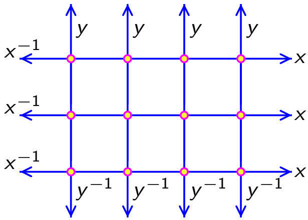
Liu studied this for separable potentials, e.g. $P(u) = \sum_i P_i(u_i)$ $u = (u_1, \ldots, u_d)$ (or a coarser decomposition)

He showed a number of rigidity results, all on the grid graph.

In $\ell_z(\Gamma)$, we may consider $f \in \mathbb{C}^W$.

$$H_z f(v) = P(v)f(v) + \sum_{(u,v)\in E} c_{u,v}(f(v) - z_{u,v}f(u))\,,$$

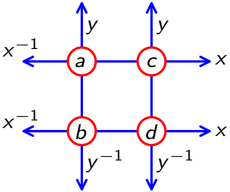
where $z_{u,v} = 1$, unless the edge leaves W, and then it is as indicated.



Example $q_1 = q_2 = 2$

Absorb the energy λ and $\sum_u c_{u,v}$ into $P(v)$.

$$-H = \begin{pmatrix} -a & 1+\frac{1}{y} & 1+\frac{1}{x} & 0 \\ 1+y & -b & 0 & 1+\frac{1}{x} \\ 1+x & 0 & -c & 1+\frac{1}{y} \\ 0 & 1+x & 1+y & -d \end{pmatrix}$$

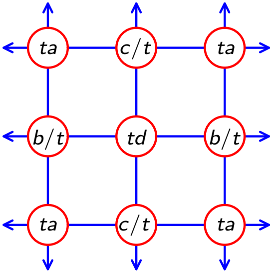


$F(x,y) = \det H \ \left(= F(x^{-1}, y^{-1})\right)$ is

$y^2 - 2x^{-1}y - y(ab+cd) - 2xy$

$+abcd - 2(ab+cd+ac+bd) + 4$

$-x(ac+bd) + x^2 + \cdots$ (symmetric)

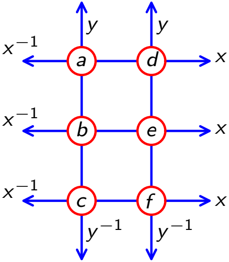


Fermi isospectral potentials Q s.t. $F_Q = F$ form a curve with 4 components.
For each of 4 choices of fundamental domain and any $t \in \mathbb{C}^\times$, we get Fermi isospectral potential.

Example $q_1 = 2$ and $q_2 = 3$

The operator $-H$ is

$$\begin{pmatrix} -a & 1 & \frac{1}{y} & 1+\frac{1}{x} & 0 & 0 \\ 1 & -b & 1 & 0 & 1+\frac{1}{x} & 0 \\ y & 1 & -c & 0 & 0 & 1+\frac{1}{x} \\ 1+x & 0 & 0 & -d & 1 & \frac{1}{y} \\ 0 & 1+x & 0 & 1 & -e & 1 \\ 0 & 0 & 1+x & y & 1 & -f \end{pmatrix}$$



The coefficients of $F(x,y) = \det H$ are affine combinations of

$a+b+c+d+e+f$, $ad+be+cf$, $abc+def$, $adbe+adcf+becf$, and $abcdef$.

As there are five (invariant) polynomials and six parameters, sets of Fermi isospectral potentials have dimension at least 1.

In fact, they form a curve of degree 72.

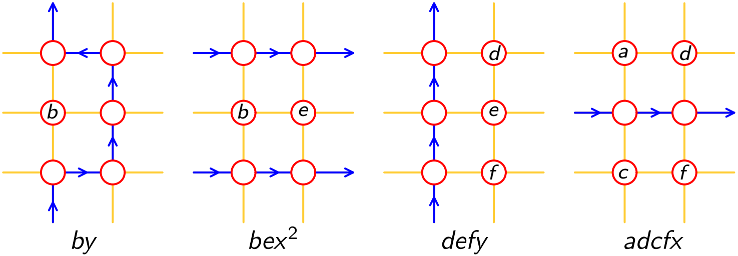
Curves of Fermi Isospectral Potentials?

Theorem. For any q_1, q_2 both ≥ 2, sets of Fermi isospectral potentials have dimension at least one.

Conjecture. For general potentials, this is a curve.

Our method is to study the coefficients as polynomials in the potential, seeking to show they are algebraically independent.

Very helpful is a graphical notation for terms in the expansion of the determinant; The potential appears only through fixed points.



This idea appeared in Harrison's talk

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