Hyperbolic Footballs

My Favorite Adapted Math Circle Topic

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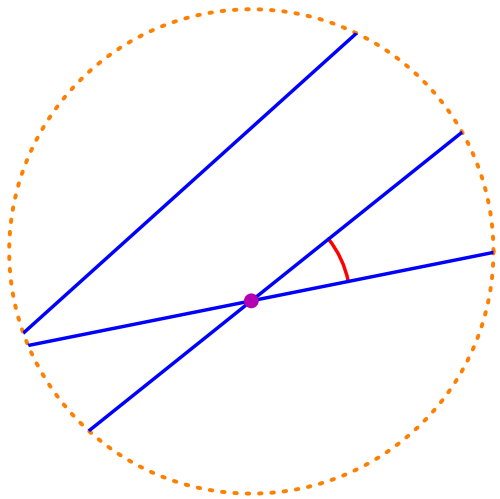
Math 467: Modern Geometry

This capstone course for future secondary mathematics teachers covers Euclidean and non-Euclidean geometry, from a historical perspective.

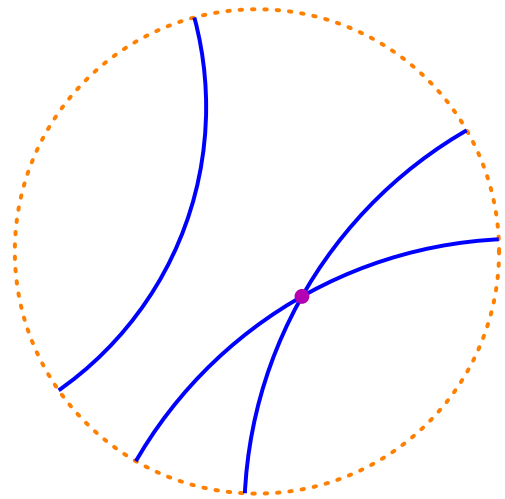
Problem: Models for the non-Euclidean Plane.

Beltrami



Not conformal (angles)

Poincaré



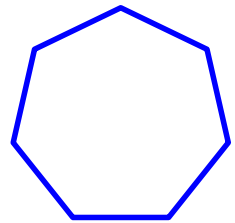
Lines not straight

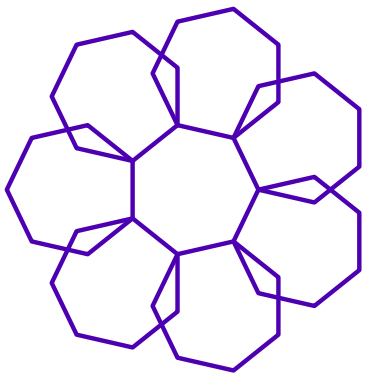
In both models, the length is weird, as lines are evidently finite. Neither is intuitive, even for the instructor.

More 467 & 367

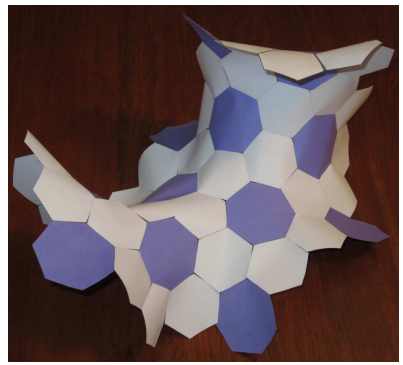
Problem Solution? Physical models.

My wife was teaching 367 (geometry for primary teachers). She had the students cut out heptagons and tape them together along their edges.



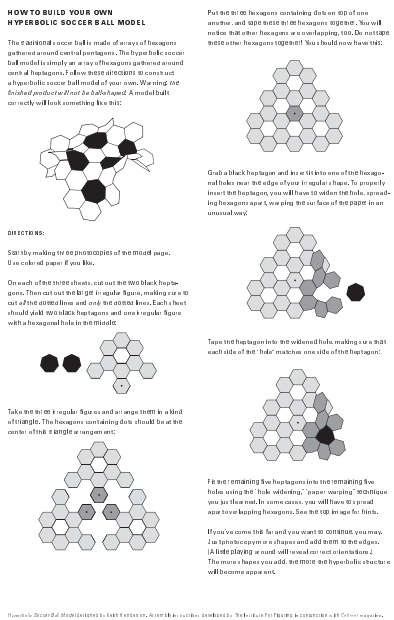


Problem: Too much cutting and curvature!



Tried something else, but too much cutting:

Googled a solution....
Found a model designed by D. Henderson in *Cabinet Magazine*.



HOW TO BUILD YOUR OWN
HYPERBOLIC SOCCER BALL MODEL

The traditional soccer ball is made of arrays of hexagons gathered around central pentagons. The hyperbolic soccer ball model is simply an array of hexagons gathered around central heptagons. Follow these directions to construct a hyperbolic soccer ball model of your own. Warning: *the finished product will not be ballshaped.* A model built correctly will look something like this:

DIRECTIONS:

Start by making several photocopies of the model page. Use colored paper if you like.

On each of five index sheets, cut out the two black heptagons. Then cut out the large irregular figure, making sure to cut *along* the dotted lines and *only* the dotted lines. Each sheet should yield two black heptagons and one irregular figure with a hexagonal hole in the middle.

Take the three irregular figures and arrange them in a kind of triangle. The hexagons containing dots should be at the center of this triangle arrangement:

Put the three hexagons containing dots on top of one another, and tape these three hexagons together. You will notice that other hexagons are overlapping, too. Do not tape these other hexagons together! You should now have this:

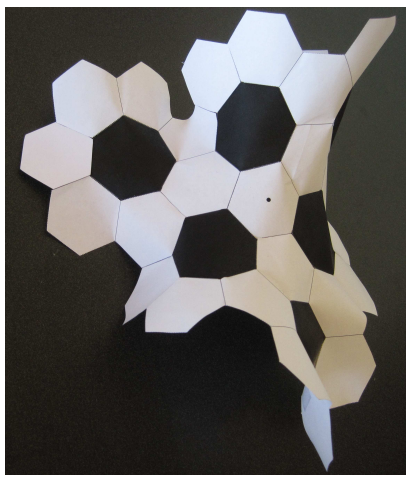
Grab a black heptagon and insert it into one of the hexagonal holes near the edges of your irregular figure. To properly insert the heptagon, you will have to widen the hole, spreading hexagons apart, warping the surface of the paper in an unusual way:

Tape the heptagon into the widened hole, making sure that each side of the "hole" matches one side of the heptagon:

Fit the remaining five heptagons into the remaining five holes using the "hole widening," "paper warping" technique you just learned. In some cases, you will have to spread apart overlapping hexagons. See the top image for hints.

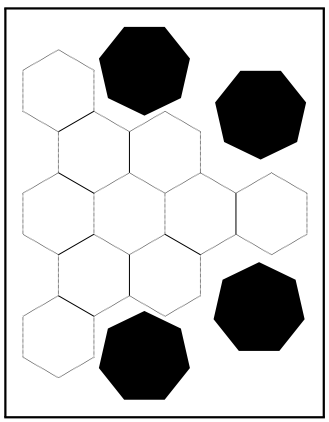
If you've come this far and you want to continue, you may. Just photocopy more shapes and add them to the edges. (A little playing around will reveal correct orientations.) The more shapes you add, the more the hyperbolic structure will become apparent.

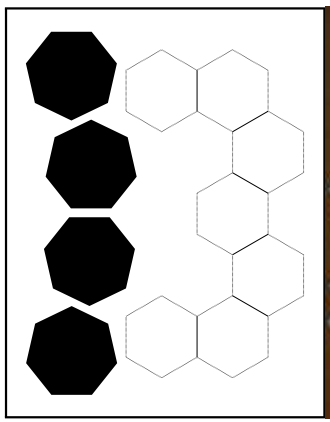
Hyperbolic Soccer Ball Model designed by Keith Henderson. Assembly instructions developed by The Institute For Figuring in conjunction with Cabinet magazine.

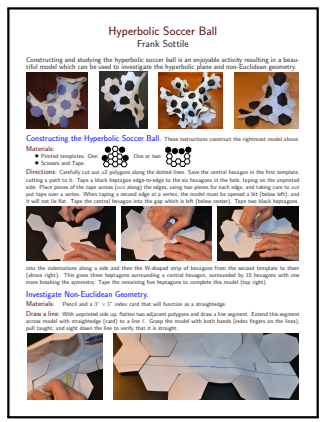


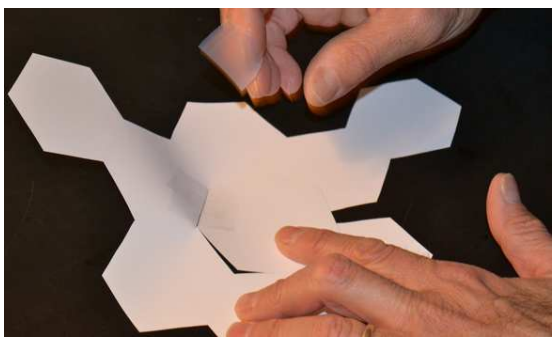
The Hyperbolic Football

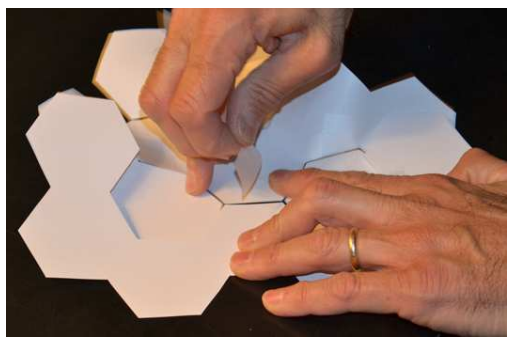
After these lessons, designed templates (programmed in postscript) and created assembly instructions and a handout.

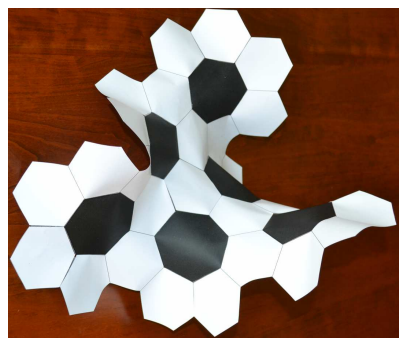








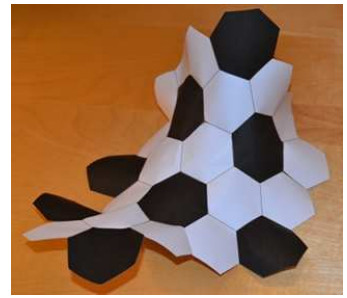




Used this several times in class. Remark from graduating senior:

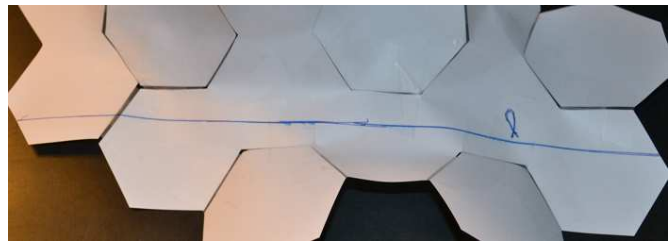
"My education ended as it began, with scissors, paper, and tape."

Non-Euclidean Geometry



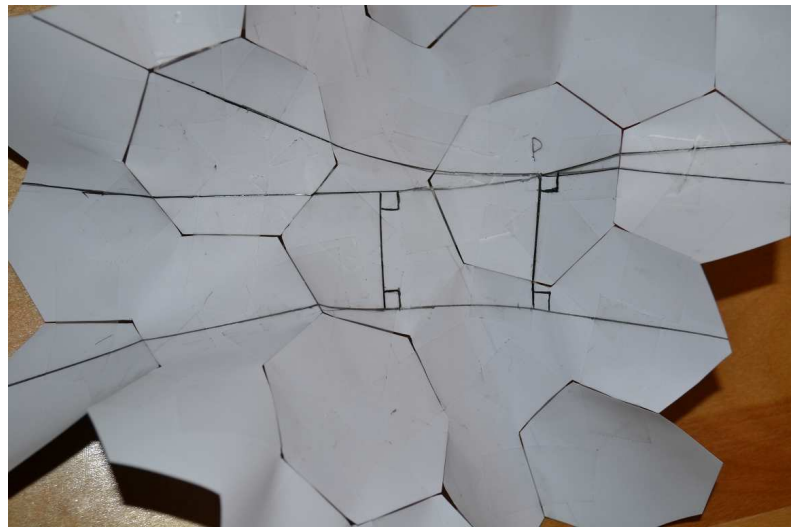
While the model is beautiful, the mathematics is more so.

I designed an activity using the real estate on the back to show:



- Every two points determine a line.
- Lines may be extended.
- Lines with a common perpendicular are parallel.

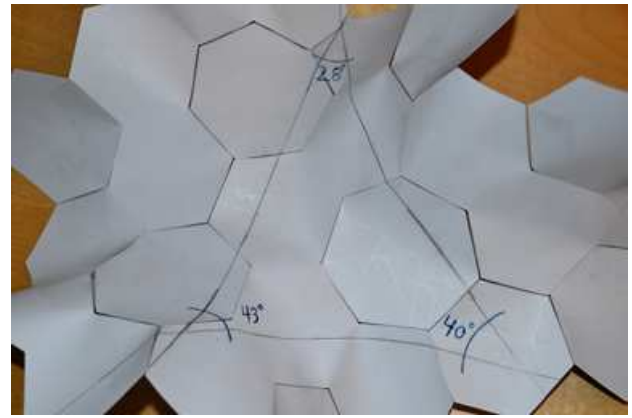
- Playfair's axiom fails. ⟹

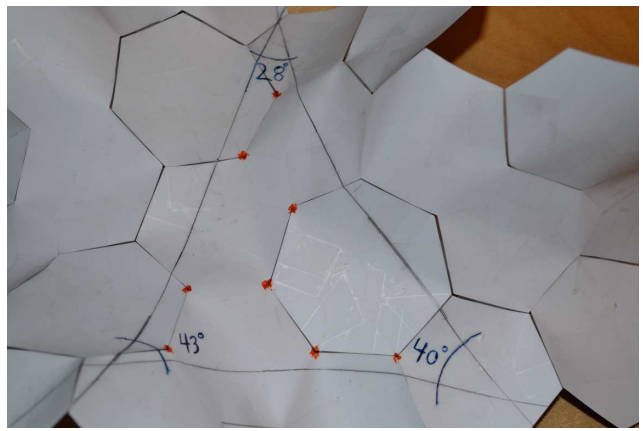


(Aficionados should note
the Lambert quadrilateral.)

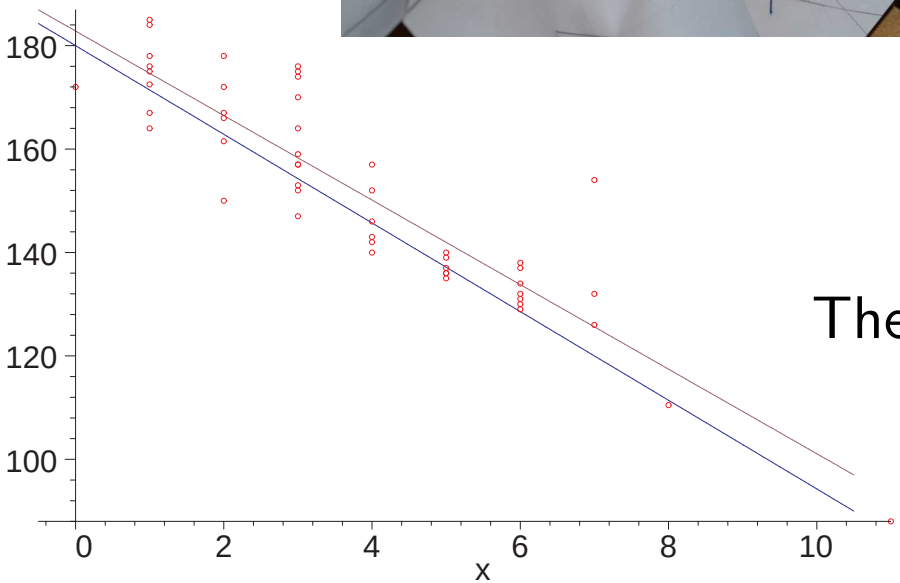
There is more: Triangles

The angle sum of triangles is evidently not 180°, with bigger triangles having smaller angle sums.





Students investigate, using an analog measure of angle sum, and # internal vertices as a proxy for area.



The plot is interesting.

As a Math Circle....

- In a class, the whole activity takes 75 minutes. It is better if students cut their templates before class.
- Nearly perfect for a 1.5-2 hour Math circle—there are many off-ramps. (ages 12 and up, boys have dexterity problems.)
- I've done this eight times in a class, other faculty borrow it, and I have used it in **23** circles in the US, Canada, and Nigeria.
- Other circle topics work in a class.

I have some materials, several designs, detailed of instructions, and activity description on my website.

University of Ilorin, 2012.