Welschinger signs and the Wronski map

Computational Real Algebraic Geometry

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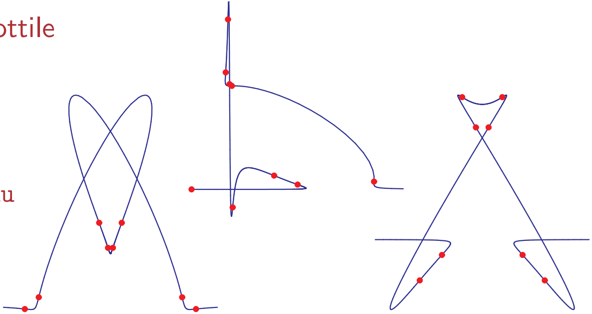
Frank Sottile

Texas A&M University
sottile@math.tamu.edu

With Tom Brazelton



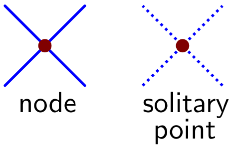
Nodes and Signs of Rational Plane Curves

An irreducible plane curve C of degree d has arithmetic genus $\binom{d-1}{2}$.

$\Rightarrow$ when C is rational ($g = 0$), it necessarily has singularities.

If C is also general, it has $\binom{d-1}{2}$ ordinary double points ()

Real curves have three types of ordinary double points:



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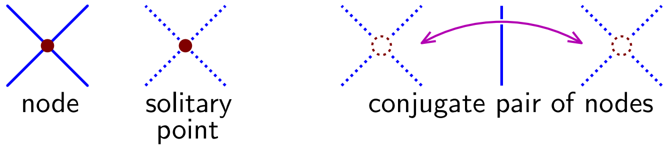

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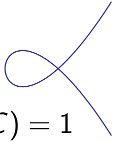
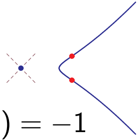
Welschinger's Lower Bound



$w(C) = 1$



$w(C) = -1$

c. 1990, Kontsevich gave a formula for the number N_d of rational curves through $3d - 1$ general points in $\mathbb{P}^2$.

Welschinger c. 2002: *If each of the $3d - 1$ points are real, then*

$$\sum_{C \text{ real}} w(C)$$

is independent of the choice of $3d - 1$ general real points.

IKS: This sum, W_d is at least $\frac{d!}{3}$ and $\lim_{d\to\infty} \frac{\log W_d}{\log N_d} = 1$.

d	1	2	3	4	5
N_d	1	1	12	620	87304
W_d	1	1	8	240	18264

Parametrized Rational Curves From Grassmannians

Let $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^d = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d))^*) = \mathbb{P}(V)$ be the rational normal curve.

A codimension $k{+}1$ plane H in $\mathbb{P}^d$ ($H \in \mathbb{G}$) is the centre of a linear projection $\pi_H\colon \mathbb{P}^d \to \mathbb{P}(V/H) = \mathbb{P}^k$, and this induces a map $\gamma_H\colon \mathbb{P}^1 \to \mathbb{P}^k$, which is a parametrized rational curve of degree d.

Singularities of γ_H correspond to the interaction of H with γ.

For example, a *flex* (first k derivatives dependent) at $\gamma_H(s)$ corresponds to H meeting the osculating k-plane $F_k(s)$ to γ at $s \in \mathbb{P}^1$.

Cusps and higher order *ramification* of γ_H correspond to Schubert conditions that H satisfies with respect to the flag $F_\bullet(s)$ osculating γ at $\gamma(s)$.

(This is classical, going back to 19th c. and used by Eisenbud-Harris in the 1980's.)

The Wronski Map

Given $H \in \mathbb{G}$, we get the rational curve $\gamma_H = (f_0(s,t), \ldots, f_k(s,t))$ (f_i is homogeneous of degree d). The *Wronskian* of H is

$$\mathrm{W}r(H) \;:=\; \det\left(\frac{\partial^a}{\partial s^a}\frac{\partial^b}{\partial t^b} f_i(s,t)\right)_{a+b=k}^{i=0\ldots,k} \;\in\; \mathbb{P}(H^0(\mathcal{O}(N))^*)\,.$$

Here, $N := (k{+}1)(d{-}k) = \dim \mathbb{G}$.

Zeroes of $\mathrm{W}r(H) \longleftrightarrow$ flexes of γ_H.

This *Wronski map* $\mathbb{G} \ni H \mapsto \mathrm{W}r(H)$ is the restriction to $\mathbb{G}$ of a linear projection

$$\mathbb{P}(\wedge^{k+1} H^0(\mathcal{O}(d))^*) \;\longrightarrow\; \mathbb{P}(H^0(\mathcal{O}(N))^*) = \mathbb{P}^N\,.$$

Easy fact: This is a finite map $\mathbb{G} \to \mathbb{P}^N$ of degree

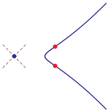
$$\deg \mathbb{G} \;=\; \frac{1!2!\cdots(d{-}k{-}1)!\cdot N!}{k!(k{+}1)!\cdots(d{-}1)!}\,.$$

Maximally Inflected Curves

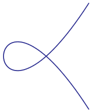
Theorem. (MTV) *If* $f \in \mathbb{P}^N$ *is hyperbolic (all roots real), then* $\mathrm{W}r^{-1}(f) \subset \mathbb{G}_{\mathbb{R}}$. *If* f *has distinct roots, it is a regular value of* $\mathrm{W}r$.

(If $\mathrm{W}r(H)$ has all roots real, then H is necessarily real.)

Definition. (Kharlamov-S.) If $H \in \mathbb{G}$ and $\mathrm{W}r(H)$ is hyperbolic, then γ_H is *maximally inflected* in that all flexes occur at real points.

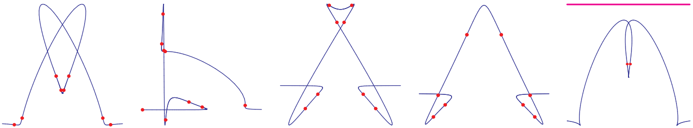


Maximally inflected



Not maximally inflected

These curves are beautiful; here are a few quintics.



New Conjectured Reality

Restricting $\mathrm{W}r$ to the big cell of $\mathbb{G}_{\mathbb{R}}$ gives a proper map $\mathrm{W}r\colon \mathbb{R}^N \to \mathbb{R}^N$ (=monic real polynomials of degree N).

Eremenko and Gabrielov computed its degree for all k and d (formula omitted).

Curious Conjecture: (Brazelton-S.) Fix $k = 2$. When γ_H is maximally inflected with only flexes (e.g. $\mathrm{W}r(H)$ is hyperbolic with simple roots), then $\deg_H \mathrm{W}r = (-1)^d w(\gamma_H)$:

Sign of the Wronski map at H = Welschinger sign of curve γ_H.

This is easily proven when $d = 4$, and there is significant evidence for $d = 5, 6$. Computations, even for $d = 6$ are challenging.

Obvious generalizations do not appear to hold.

Even More Reality, Experimentally

When $k = 2$, $d = 5$, and f is hyperbolic with simple roots, then $\#\mathrm{W}r^{-1}(f) = 42$.

Define $S(j)_f := \#\{H \in \mathrm{W}r^{-1}(f) \mid \gamma_H \text{ has j solitary points}\}$.

In each of $\gtrsim 10^6$ examples, we find that
$S_5 = (S(j)_f \mid j = 0 \ldots 6) = (0, 0, 0, 12, 18, 9, 3)$.

When $d = 4$, we have $\#\mathrm{W}r^{-1}(f) = 5$, and it is a result of Kharlamov-S. that $S_4 = (0, 0, 3, 2)$ for any f.

$d = 6$, we have $\#\mathrm{W}r^{-1}(f) = 462$, and in about 200 challenging examples, we find that

$S_6 = (0, 0, 0, 0, 5, 132, 132, 88, 39, 12, 4)$.

Another Reality Conjecture

Other ramifications may be imposed on plane curves: E.g. in a local parameter, the curve is $s \mapsto (s^{1+b}, s^{1+a})$ with $a \geq b \geq 0$. (A simple flex is $(a, b) = (1, 0)$.)

This ramification has order $a + b$, and the sum of all local ramifications of a curve is $N = 3(d-2)$.

Assigning ramifications to points of $\mathbb{RP}^1 \simeq S^1$ gives a necklace, e.g. (cusp,flex,cusp,flex,cusp,flex)$\neq$(flex,flex,flex,cusp,cusp,cusp).

Conjecture. For a given necklace ν of ramification, the vector

$$(\#\text{curves with given ramification and } i \text{ solitary points} \mid i)$$

is independent of the placement of the points of ramification.

This has been tested thousands of times for all ramification when $d = 5$ and many times for $d = 6$.