Surface Divergences and Boundary Energies in the Casimir Effect

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Introduction

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while other divergences are associated with the surface. The interpretation of these has been quite controversial. Particularly mysterious is the contradiction between finite total self-energies (Bernasconi, Graf, and Hasler) and surface divergences in the local energy density.

Gluon and quark condensates

Due to zero-point fluctuations, confinement of gluon and quark fields in the bag model (presumably a first-mock-up of QCD) will result in a nonzero expectation value for the squared field strength,

$$\langle G^2(r)\rangle = \frac{1}{4\pi^2 a}\frac{1}{r^2}\frac{d}{dr}\sum_{l=1}^{\infty}(2l+1)2\int_0^\infty dx\, e^{ix\delta}\frac{s_l(xr/a)s_l'(xr/a)}{s_l(x)s_l'(x)},$$

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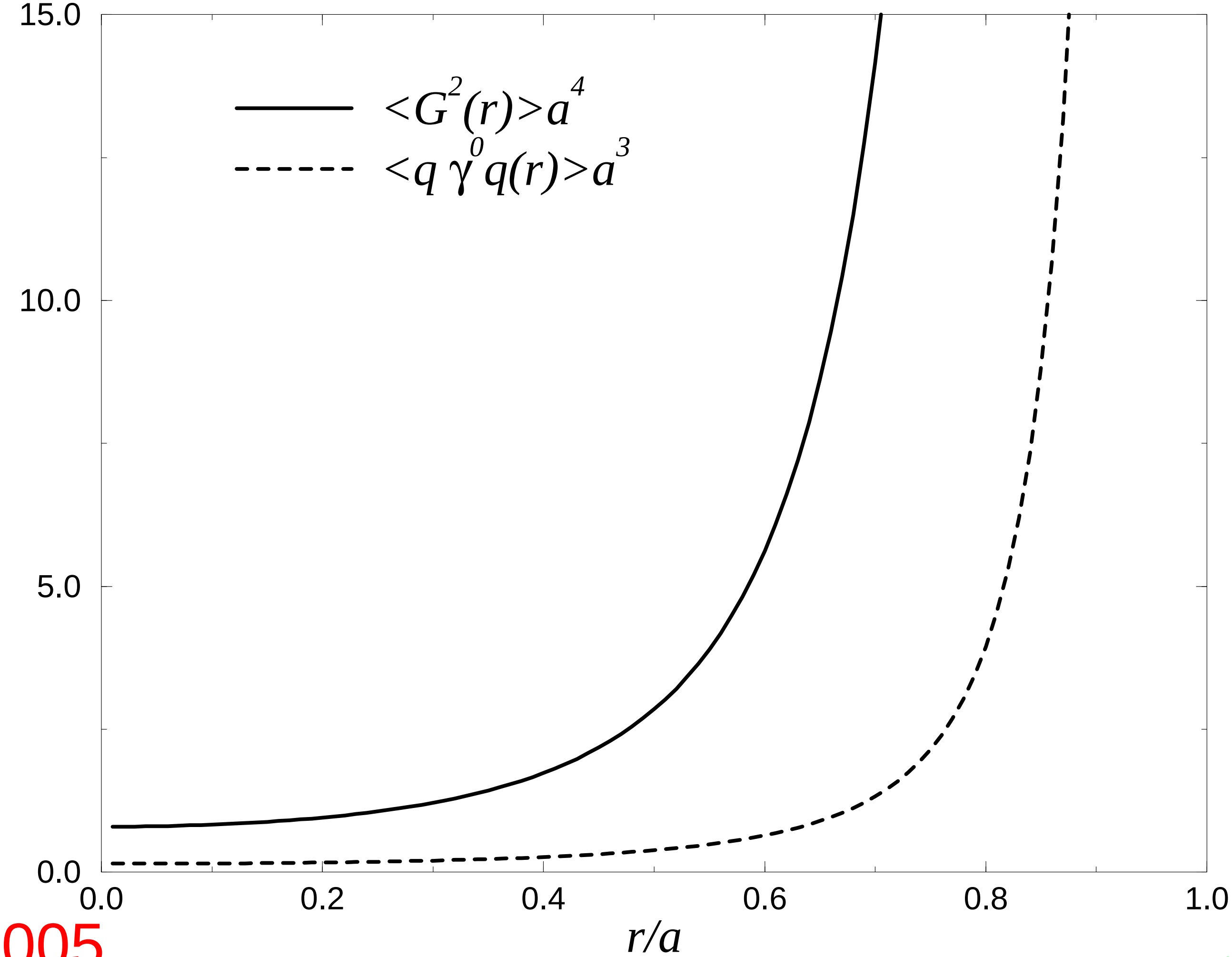
Similarly, we obtain the following expression for the quark condensate

$$\langle\bar{q}q(r)\rangle = -\frac{1}{4\pi^2 a r^2}\sum_{l=0}^{\infty}2(l+1)\int_{-\infty}^{\infty} dx\, e^{ix\delta}\frac{s_l^2(xr/a)+s_{l+1}^2(xr/a)}{s_l^2(x)+s_{l+1}^2(x)}.$$

Here s_l and e_l are modified spherical Bessel functions.

Condensate Graphs



Condensate Graphs Captions

- Magnitude of the quark and gluon condensates as functions of the distance from the center of the bag. The values shown are for a single color, and, for the quarks, for a single flavor and helicity state.
- For further details see K. A. Milton, *The Casimir Effect* (World Scientific, 2001).

Casimir Forces on Spheres

The calculations presented here were carried out in response to the program of the MIT group. They rediscovered irremovable divergences in the Casimir energy for a circle first discovered by Sen in 1980, but then found divergences in the case of a spherical surface, thereby casting doubt on the validity of the Boyer calculation. Some of their results, as we shall see, are spurious, and the rest had been earlier discovered by the Leipzig group. However, their work has been valuable in sparking new investigations of the problems of surface energies and divergences.

δ-function Potential

We now carry out a calculation of the zero-point energy in three spatial dimensions, with a radially symmetric singular background ($[\lambda] = \mathrm{L}^{-1}$)

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\lambda\delta(r-a)\phi^2(x),$$

which would correspond to a Dirichlet shell in the limit $\lambda \to \infty$. The time-Fourier transformed Green's function satisfies the equation ($\kappa^2 = -\omega^2$)

$$\left[-\nabla^2 + \kappa^2 + \lambda\delta(r-a)\right] G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

We write G in terms of a reduced Green's function

$$G(\mathbf{r}, \mathbf{r}') = \sum_{lm} g_l(r, r') Y_{lm}(\Omega) Y^*_{lm}(\Omega'),$$

where g_l satisfies

$$\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{l(l+1)}{r^2}+\kappa^2+\lambda\delta(r-a)\right]g_l(r,r')=\frac{1}{r^2}\delta(r-r').$$

We solve this in terms of modified Bessel functions, $I_\nu(x)$, $K_\nu(x)$, where $\nu=l+1/2$, which satisfy the Wronskian condition

$$I'_\nu(x)K_\nu(x)-K'_\nu(x)I_\nu(x)=\frac{1}{x}.$$

The solution is obtained by requiring continuity of g_l at each singularity, r' and a, and the appropriate discontinuity of the derivative. Inside the sphere we then find ($0<r,r'<a$)

$$g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{\lambda}{\kappa}e_l^2(\kappa a)\frac{s_l(\kappa r)s_l(\kappa r')}{1+\frac{\lambda}{\kappa}s_l(\kappa a)e_l(\kappa a)}\right].$$

Here we have introduced the modified Riccati-Bessel functions,

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_{l+1/2}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}}K_{l+1/2}(x).$$

Note that this reduces to the expected result, vanishing as $r \to a$, in the limit of strong coupling:

$$\lim_{\lambda\to\infty} g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{e_l(\kappa a)}{s_l(\kappa a)}s_l(\kappa r)s_l(\kappa r')\right].$$

When both points are outside the sphere, $r, r' > a$, we obtain a similar result:

$$g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{\lambda}{\kappa}s_l^2(\kappa a)\frac{e_l(\kappa r)e_l(\kappa r')}{1+\frac{\lambda}{\kappa}s_l(\kappa a)e_l(\kappa a)}\right].$$

which similarly reduces to the expected result as $\lambda \to \infty$.

Pressure on sphere

Now we want to get the radial-radial component of the stress tensor to extract the pressure on the sphere, which is obtained by applying the operator

$$\partial_r\partial_{r'} - \frac{1}{2}(-\partial^0\partial'^0 + \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}') \to \frac{1}{2}\left[\partial_r\partial_{r'} - \kappa^2 - \frac{l(l+1)}{r^2}\right]$$

to the Green's function, where in the last term we have averaged over the surface of the sphere. In this way we find, from the discontinuity of $\langle T_{rr}\rangle$ across the $r=a$ surface, the net stress

$$\mathcal{S} = \frac{\lambda}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\, \frac{(e_l(x)s_l(x))' - \frac{2e_l(x)s_l(x)}{x}}{1+\frac{\lambda a e_l(x)s_l(x)}{x}}.$$

The same result can be deduced by computing the total energy. The free Green's function, evidently, makes no significant contribution to the energy, for it gives a term independent of the radius of the sphere, a, so we omit it. The remaining radial integrals are simply

$$\int_0^x dy\, s_l^2(y) = \frac{1}{2x}\left[\left(x^2 + l(l+1)\right) s_l^2 + x s_l s_l' - x^2 s_l'^2\right],$$
$$\int_x^\infty dy\, e_l^2(y) = -\frac{1}{2x}\left[\left(x^2 + l(l+1)\right) e_l^2 + x e_l e_l' - x^2 e_l'^2\right],$$

where all the Bessel functions on the right-hand-sides of these equations are evaluated at x. Then using the Wronskian, we find that the Casimir energy is

$$E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty} dx\, x\, \frac{d}{dx}\ln\left[1+\lambda a I_\nu(x)K_\nu(x)\right].$$

If we differentiate with respect to a, with λ fixed, we immediately recover the force. This expression, upon integration by parts, coincides with that given recently by Barton, and was first analyzed in detail by Scandurra.

Strong coupling

For strong coupling, the above energy reduces to the well-known expression for the Casimir energy of a massless scalar field inside and outside a sphere upon which Dirichlet boundary conditions are imposed, that is, that the field must vanish at $r=a$:

$$\lim_{\lambda\to\infty} E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\, x\,\frac{d}{dx}\ln\left[I_\nu(x)K_\nu(x)\right],$$

($\nu = l+1/2$) because multiplying the argument of the logarithm by a power of x is without effect, corresponding to a contact term.

Boyer Coefficients

This may be evaluated numerically:

$$E^{\mathrm{TE}} = \frac{0.002817}{a},$$

which is much smaller than the Boyer result for electrodynamics:

$$E^{\mathrm{EM}} = \frac{0.04618}{a},$$

although both are repulsive.

Weak coupling

The opposite limit is of interest here. The expansion of the logarithm is immediate for small λ. The first term, of order λ, is evidently divergent, but irrelevant, since that may be removed by renormalization of the tadpole graph. In contradistinction to the claim of of the MIT group the order λ^2 term is finite. That term is

$$E^{(\lambda^2)} = \frac{\lambda^2 a}{4\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\, x\frac{d}{dx}[I_{l+1/2}(x)K_{l+1/2}(x)]^2.$$

The sum on l can be carried out using a trick due to Klich: The sum rule

$$\sum_{l=0}^{\infty}(2l+1)e_l(x)s_l(y)P_l(\cos\theta) = \frac{xy}{\rho}e^{-\rho},$$

where $\rho = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, is squared, and then integrated over θ, according to

$$\int_{-1}^{1} d(\cos\theta) P_l(\cos\theta) P_{l'}(\cos\theta) = \delta_{ll'} \frac{2}{2l+1}.$$

In this way we learn that

$$\sum_{l=0}^{\infty} (2l+1) e_l^2(x) s_l^2(x) = \frac{x^2}{2} \int_0^{4x} \frac{dw}{w} e^{-w}.$$

Although this integral is divergent, because we did not integrate by parts, that divergence does not contribute:

$$E^{(\lambda^2)}=\frac{\lambda^2 a}{4\pi}\int_0^\infty dx\,\frac{1}{2}x\,\frac{d}{dx}\int_0^{4x}\frac{dw}{w}e^{-w}=\frac{\lambda^2 a}{32\pi},$$

which is exactly the result I had found earlier, based on the following formula for a hypersphere in D dimensions:

$$E_D^{(\lambda^2)}=-\frac{\lambda^2 a}{\pi}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma(D-3/2)\Gamma(1-D/2)}{2^{1+2D}[\Gamma(D/2)]^2}.$$

which exhibits poles when D is even, where the Casimir energy is known to diverge.

Divergences

However, before we wax too euphoric, we recognize that the order λ^3 term appears logarithmically divergent, just as the MIT group claimed. Suppose we subtract off the two leading terms,

$$
\begin{aligned}
E &= -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\,x\,\frac{d}{dx}\Big[\ln\left(1+\lambda a I_\nu K_\nu\right)-\lambda a I_\nu K_\nu \\
&\quad + \frac{\lambda^2 a^2}{2}(I_\nu K_\nu)^2\Big] + \frac{\lambda^2 a}{32\pi}.
\end{aligned}
$$

To study the behavior of the sum for large values of l, we can use the uniform asymptotic expansion (Debye expansion),

$$\nu \gg 1: \quad I_\nu(x)K_\nu(x) \sim \frac{t}{2\nu}\left[1 + \frac{A(t)}{\nu^2} + \frac{B(t)}{\nu^4} + \dots\right].$$

Here $x = \nu z$, and $t = 1/\sqrt{1+z^2}$. The functions A and B, etc., are polynomials in t. We now insert this into the energy expression and expand not in λ but in ν; the leading term is

$$E^{(\lambda^3)} \sim \frac{\lambda^3 a^2}{24\pi}\sum_{l=0}^{\infty}\frac{1}{\nu}\int_0^\infty \frac{dz}{(1+z^2)^{3/2}} = \frac{\lambda^3 a^2}{24\pi}\zeta(1).$$

Although the frequency integral is finite, the angular momentum sum is divergent. The appearance here of the divergent $\zeta(1)$ seems to signal an insuperable barrier to extraction of a finite Casimir energy for finite λ. The situation is different in the limit $\lambda \to \infty$.

- This divergence has been known for many years, and was first calculated explicitly in 1998 by Bordag et al., where the second heat kernel coefficient gave

$$E \sim \frac{\lambda^3 a^2}{48\pi}\frac{1}{s}, \quad s \to 0.$$

A possible way of dealing with this divergence was advocated in Scandurra. Very recently, Bordag and Vassilevich have reanalyzed such problems from the heat kernel approach. They show that this $O(\lambda^3)$ divergence corresponds to a surface tension counterterm, an idea proposed by me in 1980 in connection with the zero-point energy contribution to the bag model. Such a surface term corresponds to λ fixed, which then necessarily implies a divergence of order λ^3. Bordag argues that it is perfectly appropriate to render this divergence finite by renormalization.

Boundary layer energy

Here we show that the surface energy can be interpreted as the bulk energy of the boundary layer. We do this by considering a scalar field in $d+1+1$ dimensions interacting with the background

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{2}\phi^2\sigma,$$

where

$$\sigma(x) = \begin{cases} h, & -\frac{\delta}{2} < x < \frac{\delta}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

with the property that $h\delta = 1$.

Reduced Green's Function

The reduced Green's function satisfies

$$\left[-\frac{\partial^2}{\partial x^2}+\kappa^2+\lambda\sigma(x)\right]g(x,x')=\delta(x-x').$$

This may be easily solved in the region of the slab, $-\frac{\delta}{2}<x<\frac{\delta}{2}$, ($\kappa'=\sqrt{\kappa^2+\lambda h}$)

$$g(x,x')=\frac{1}{2\kappa'}\left\{e^{-\kappa'|x-x'|}+\frac{1}{\hat\Delta}\left[(\kappa'^2-\kappa^2)\cosh\kappa'(x+x')\right.\right.$$
$$\left.\left.+(\kappa'-\kappa)^2e^{-\kappa'\delta}\cosh\kappa'(x-x')\right]\right\},$$

$$\hat\Delta=2\kappa\kappa'\cosh\kappa'\delta+(\kappa^2+\kappa'^2)\sinh\kappa'\delta.$$

Local Energy Density

Consider the stress tensor with an arbitrary conformal term,

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi + \lambda h\phi^2) - \alpha(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2.$$

We get the following form for the energy density within the slab, $[\langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle = -iG(\mathbf{r},\mathbf{r}')]$

$$\begin{aligned} T^{00} &= \frac{2^{-d-2}\pi^{-(d+1)/2}}{\Gamma((d+3)/2)}\int_0^\infty \frac{d\kappa\,\kappa^d}{\kappa'\hat\Delta}\bigg\{(\kappa'^2-\kappa^2) \\ &\quad\times\left[(1-4\alpha)(1+d)\kappa'^2-\kappa^2\right]\cosh 2\kappa' x \\ &\quad-(\kappa'-\kappa)^2 e^{-\kappa'\delta}\kappa^2\bigg\}. \end{aligned}$$

From this we can calculate the behavior of the energy density as the boundary is approached from the inside:

$$T^{00} \sim \frac{\Gamma(d+1)\lambda h}{2^{d+4}\pi^{(d+1)/2}\Gamma((d+3)/2)} \frac{1-4\alpha(d+1)/d}{(\delta-2|x|)^d}, \qquad |x| \to \delta/2.$$

For $d=2$ for example, this agrees with the result found by Graham and Olum for $\alpha = 0$:

$$T^{00} \sim \frac{\lambda h}{96\pi^2} \frac{(1-6\alpha)}{(\delta/2-|x|)^d}, \qquad |x| \to \frac{\delta}{2}.$$

Note that, as we expect, this surface divergence vanishes for the conformal stress tensor, where $\alpha = d/4(d+1)$. (There will be subleading divergences if $d>2$.)

We can also calculate the energy density on the other side of the boundary, from the Green's function for $x, x' < -\delta/2$,

$$g(x,x') = \frac{1}{2\kappa}\left[e^{-\kappa|x-x'|} - e^{\kappa(x+x'+\delta)}(\kappa'^2-\kappa^2)\frac{\sinh\kappa'\delta}{\hat{\Delta}}\right],$$

and the corresponding energy density is given by

$$T^{00} = -\frac{d(1-4\alpha(d+1)/d)}{2^{d+2}\pi^{(d+1)/2}\Gamma((d+3)/2)}\int_0^\infty d\kappa\,\kappa^{d+1}\frac{1}{\hat{\Delta}}(\kappa'^2-\kappa^2) \times e^{2\kappa(x+\delta/2)}\sinh\kappa'\delta,$$

which vanishes if the conformal value of α is used.

The divergent term, as $x \to -\delta/2$, is just the negative of that found on the inside. This is why, when the total energy is computed by integrating the energy density, it is finite for $d < 2$, and independent of α. The divergence encountered for $d = 2$ may be handled by renormalization of the interaction potential. In the limit as $h \to \infty$, $h\delta = 1$, we recover the divergent expression for a single interface

$$\lim_{h\to\infty} E_s = \frac{1}{2^{d+2}\pi^{(d+1)/2}\Gamma((d+3)/2)} \int_0^\infty d\kappa\, \kappa^d \frac{\lambda}{\lambda + 2\kappa}.$$

Therefore, surface divergences have an illusory character.

TM Spherical Potential

Of course, the scalar model considered in the above is merely a toy model, and something analogous to electrodynamics is of far more physical relevance. There are good reasons for believing that cancellations occur in general between TE (Dirichlet) and TM (Robin) modes. Certainly they do occur in the classic Boyer energy of a perfectly conducting spherical shell, and the indications are that such cancellations occur even with imperfect boundary conditions – See Barton. Following the latter reference, let us consider the potential

$$\mathcal{L}_{\rm int} = \frac{1}{2}\lambda\frac{1}{r}\frac{\partial}{\partial r}\delta(r-a)\phi^2(x).$$

In the limit $\lambda \to \infty$ this corresponds to TM BCs.

It is then easy to find the Green's function. When both points are inside the sphere, $r, r' < a$:

$$g_l(r, r') = \frac{1}{\kappa r r'} \left[s_l(\kappa r_<) e_l(\kappa r_>) - \frac{\lambda \kappa [e_l'(\kappa a)]^2 s_l(\kappa r) s_l(\kappa r')}{1 + \lambda \kappa e_l'(\kappa a) s_l'(\kappa a)} \right],$$

and when both points are outside the sphere, $r, r' > a$:

$$g_l(r, r') = \frac{1}{\kappa r r'} \left[s_l(\kappa r_<) e_l(\kappa r_>) - \frac{\lambda \kappa [s_l'(\kappa a)]^2 e_l(\kappa r) e_l(\kappa r')}{1 + \lambda \kappa e_l'(\kappa a) s_l'(\kappa a)} \right].$$

The Casimir energy may be readily obtained:

$$E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\, x\frac{d}{dx}\ln\left[1+\frac{\lambda}{a}xe_l'(x)s_l'(x)\right].$$

In the limit $\lambda \to \infty$ the sum of the TE and TM energies reduces to the familiar expression for the perfectly conducting spherical shell (omitting the $l=0$ mode):

$$\lim_{\lambda\to\infty} E = -\frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dx\, x\left(\frac{e_l'}{e_l}+\frac{e_l''}{e_l'}+\frac{s_l'}{s_l}+\frac{s_l''}{s_l'}\right),$$

of which we gave the evaluation above.

Surface energy of λ sphere

I am currently extending the calculation given for plane surfaces to that of a finite spherical annulus. The preliminary results are quite encouraging. Again we consider the potential

$$\mathcal{L}_{\mathrm{int}} = -\frac{\lambda}{2}\phi^2\sigma(r),$$

where

$$\sigma(r) = \begin{cases} 0, & r < a_-, \\ h, & a_- < r < a_+, \\ 0, & a_+ < r. \end{cases}$$

Here $a_\pm = a \pm \delta/2$, and we set $h\delta = 1$. In the limit as $\delta \to 0$ (or $h \to \infty$) we recover the δ-function sphere considered above.

Green's function for λ sphere

A straightforward solution of the Green's function equation

$$\left(-\nabla^2 + \kappa^2 + \lambda\sigma\right) G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

in terms of the reduced Green's function

$$G(\mathbf{r}, \mathbf{r}') = \sum_{lm} g_l(r, r') Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi')$$

is as follows:

$$r, r' < a_- : \quad g_l = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<) e_l(\kappa r_>) - \frac{\tilde{\Xi}}{\Xi} s_l(\kappa r) s_l(\kappa r')\right],$$

$$r, r' > a_- : \quad g_l = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<) e_l(\kappa r_>) - \frac{\hat{\Xi}}{\Xi} e_l(\kappa r) e_l(\kappa r')\right],$$

where

$$\begin{aligned}\Xi &= [\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)] \\ &\quad\times[\kappa' e_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)s_l(\kappa' a_+)] \\ &- [\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)] \\ &\quad\times[\kappa' e_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)e_l(\kappa' a_+)].\end{aligned}$$

where

$$\begin{aligned}\Xi \;=\; & [\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)] \\ & \times[\kappa' e_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)s_l(\kappa' a_+)] \\ & -[\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)] \\ & \times[\kappa' e_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)e_l(\kappa' a_+)].\end{aligned}$$

$\tilde{\Xi}$ is obtained from Ξ by replacing $s_l(\kappa a_-) \to e_l(\kappa a_-)$,

where

$$\begin{aligned}\Xi &= [\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)] \\ &\quad \times[\kappa' e_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)s_l(\kappa' a_+)] \\ &\quad - [\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)] \\ &\quad \times[\kappa' e_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)e_l(\kappa' a_+)].\end{aligned}$$

$\tilde{\Xi}$ is obtained from Ξ by replacing $s_l(\kappa a_-) \to e_l(\kappa a_-)$, while $\hat{\Xi}$ is obtained from Ξ by replacing $e_l(\kappa a_+) \to s_l(\kappa a_+)$. Here $\kappa' = \sqrt{\kappa^2 + \lambda h}$.

Green's function in annulus

$$
\begin{aligned}
g_l \quad = \quad & \frac{1}{\kappa r r'}\Bigg\{ s_l(\kappa' r_<) e_l(\kappa' r_>) \\
& - \frac{1}{\Xi}\Bigg[[s_l(\kappa' r) e_l(\kappa' r') + s_l(\kappa' r') e_l(\kappa' r)] \\
& \times [\kappa e_l'(\kappa a_+) e_l(\kappa' a_+) - \kappa' e_l(\kappa a_+) e_l'(\kappa' a_+)] \\
& \times [\kappa s_l'(\kappa a_-) s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-) s_l'(\kappa' a_-)] \\
& - s_l(\kappa' r') s_l(\kappa' r) [\kappa e_l'(\kappa a_+) e_l(\kappa' a_+) - \kappa' e_l(\kappa a_+) e_l(\kappa' a_+)] \\
& \times [\kappa s_l'(\kappa a_-) e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-) e_l'(\kappa' a_-)] \\
& - e_l(\kappa' r') e_l(\kappa' r) [\kappa e_l'(\kappa a_+) s_l(\kappa' a_+) - \kappa' e_l(\kappa a_+) s_l(\kappa' a_+)] \\
& \times [\kappa s_l'(\kappa a_-) s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-) s_l'(\kappa' a_-)] \Bigg] \Bigg\}
\end{aligned}
$$

Surface Divergences

We can calculate the local energy density from the stress tensor:

$$T^{00} = \frac{1}{2}\left[\partial^0\phi\partial^0\phi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \lambda\sigma\phi^2\right] - \alpha\nabla^2\phi^2$$

where the conformal value is $\alpha = 1/6$. In the thin-shell limit

$$\frac{\hat{\Xi}}{\Xi} \to \frac{\frac{\lambda}{\kappa}s_l^2(\kappa a)}{1+\frac{\lambda}{\kappa}e_l(\kappa a)s_l(\kappa a)},$$

we obtain using the UAE an approximate expression for the Green's function at coincident points: ($r > a$)

$$G(\mathbf{r},\mathbf{r}) \sim -\frac{i}{8\pi^2 r^2}\int_0^\infty dz\, t\sum_{l=0}^\infty \frac{\nu e^{-2\nu[\eta(z)-\eta(za/r)]}}{1+2\nu/\lambda a t(za/r)}.$$

For $\lambda \to \infty$, the l sum gives

$$G(\mathbf{r}, \mathbf{r}') = \frac{i}{32\pi^2 r^2} \int_0^\infty dz\, t \frac{\coth[\eta(z) - \eta[za/r)]}{\sin[\eta(z) - \eta(za/r)]},$$

$$\eta(z) = \sqrt{1+z^2} + \log \frac{z}{1+\sqrt{1+z^2}}, \quad t = (1+z^2)^{-1/2}.$$

As $r \to a$, is it easy to show ($\lambda \to \infty$)

$$G(\mathbf{r}, \mathbf{r}) \sim -\frac{i}{32\pi^2} \frac{1}{(r-a)^2}.$$

A similar argument shows that the energy density has the expected $(r-a)^{-4}$ divergence.

Annulus energy

In the limit of $h \to \infty$ for the region of the annulus, $a_- < r, r' < a_+$,

$$g_l \quad \to \quad -\frac{1}{2\kappa r r'} \frac{e_l(\kappa a) s_l(\kappa a)}{1 + \frac{\lambda}{\kappa} e_l(\kappa a) s_l(\kappa a)} \Bigg[\cosh \sqrt{\lambda h}(r - r') - \cosh \sqrt{\lambda h}(r + r' - a_+ - a_-) \Bigg].$$

In the thin shell limit this leads to a nearly constant energy density in the annulus ($\delta \to 0$)

$$\langle t^{00} \rangle = -i\lambda h(1 - 4\alpha) \frac{1}{2\kappa a^2} \frac{e_l(\kappa a) s_l(\kappa a)}{1 + \frac{\lambda}{\kappa} e_l(\kappa a) s_l(\kappa a)}.$$

So for the conformal value $\alpha = 1/6$ we obtain the following annulus energy

$$E_A = \frac{\lambda}{6\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx \frac{I_{l+1/2}(x)K_{l+1/2}(x)}{1+\lambda a I_{l+1/2}(x)K_{l+1/2}(x)}.$$

If this is expanded to order λ^3 we identify the corresponding term in the δ-potential Casimir energy as the surface energy!

$$E_C = \frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \log\left(1+\lambda a I_{l+1/2}(x)K_{l+1/2}\right).$$

Thus,

$$E_C - E_A = \text{finite.}$$

Dielectric Spheres

The Casimir self-stress on a uniform dielectric sphere was first worked out by me in 1979. It was generalized to the case when both electric permittivity and magnetic permeability are present in 1997. The result for the pressure, ($x = \sqrt{\varepsilon\mu}|y|$, $x' = \sqrt{\varepsilon'\mu'}|y|$ where ε', μ' are the interior, and ε, μ are the exterior, values of the permittivity and the permeability)

$$P = -\frac{1}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{iy\delta}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\Bigg\{x\frac{d}{dx}\ln D_l$$

$$+2x'[s_l'(x')e_l'(x') - e_l(x')s_l''(x')] - 2x[s_l'(x)e_l'(x) - e_l(x)s_l''(x)]\Bigg\}$$

where the “bulk” pressure has been subtracted, and

$$D_l = [s_l(x')e_l'(x) - s_l'(x')e_l(x)]^2 - \xi^2[s_l(x')e_l'(x) + s_l'(x')e_l(x)]^2,$$

with the parameter ξ being

$$\xi = \frac{\sqrt{\frac{\varepsilon'}{\epsilon}\frac{\mu}{\mu'}} - 1}{\sqrt{\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}} + 1},$$

and δ is the temporal regulator. This result is obtained either by computing the radial-radial component of the stress tensor, or from the total energy.

In general, this result is divergent. However, consider the special case $\sqrt{\epsilon\mu} = \sqrt{\epsilon'\mu'}$, that is, when the speed of light is the same in both media. Then $x = x'$ and the Casimir energy reduces to

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\delta} \sum_{l=1}^{\infty}(2l+1)x\frac{d}{dx}\ln[1-\xi^2((s_l e_l)')^2],$$

where

$$\xi = \frac{\mu-\mu'}{\mu+\mu'} = -\frac{\varepsilon-\varepsilon'}{\varepsilon+\varepsilon'}.$$

If $\xi = 1$ we recover the case of a perfectly conducting spherical shell, treated above. In fact E is finite for all ξ.

Dilute Limit

Of particular interest is the dilute limit, where

$$E \approx \frac{5\xi^2}{32\pi a} = \frac{0.099\,4718\xi^2}{2a}, \quad \xi \ll 1.$$

It is remarkable that the value for a spherical conducting shell, for which $\xi = 1$, is only 7% lower, which as Klich remarks, is accounted for nearly entirely by the next term in the small ξ expansion.

There is another dilute limit which is also quite surprising. For a purely dielectric sphere ($\mu = 1$) the leading term in an expansion in powers of $\varepsilon - 1$ is finite:

$$E = \frac{23}{1536\pi}\frac{(\varepsilon-1)^2}{a} = (\varepsilon-1)^2\frac{0.004\,767}{a}. \qquad \text{(-15)}$$

This result coincides with the sum of van der Waals energies of the material making up the ball as Ng and I showed in 1998. The term of order $(\varepsilon - 1)^3$ is divergent. The establishment of this result was the death knell for the Casimir energy explanation of sonoluminescence.

Dielectric Cylinders

The fundamental difficulty in cylindrical geometries is that there is in general no decoupling between TE and TM modes. Progress in understanding has therefore been much slower in this regime. It was only in 1981 that it was found that the electromagnetic Casimir energy of a perfectly conducting cylinder was attractive, the energy per unit length being

$$\mathcal{E}_{\rm em,cyl} = -\frac{0.01356}{a^2},$$

for a circular cylinder of radius a. The corresponding result for a scalar field satisfying Dirichlet boundary conditions of the cylinder is repulsive,

$$\mathcal{E}_{\rm D,cyl} = \frac{0.000606}{a^2}.$$

These ideal limits are finite, but, as with the spherical geometry, less ideal configurations have unremovable divergences. For example, a cylindrical δ-shell potential, as described earlier, has divergences (in third order). And it is expected that a dielectric cylinder will have a divergent Casimir energy, although the coefficient of $(\varepsilon - 1)^2$ will be finite for a dilute dielectric cylinder, corresponding to a finite van der Waals energy between the molecules that make up the material.

Casimir Energy of Dilute Cylinder

In fact, a calculation of the renormalized van der Waals energy for a dilute dielectric cylinder is zero, as is the Casimir energy for a cylinder for which the speed of light is the same inside and out, because $\varepsilon\mu = 1$.
I have just completed a calculation of the Casimir energy for a dielectric cylinder with my graduate student Ines Cavero-Pelaez (hep-th/0412135). The order $(\varepsilon - 1)^2$ term in the pressure is given by the expression

$$P = \frac{(\varepsilon - 1)^2}{16\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^\infty dy\, y^4 g_m(y),$$

where g_m is given by a rather complicated expression involving four modified Bessel functions (from which is subtracted the expression corresponding to the bulk contribution, if either the dielectric, or vacuum, filled all space, involving two Bessel functions). This expression, in fact may be straightforwardly evaluated numerically using uniform asymptotic expansions, with the unsurprising but still remarkable result

$$P = 0 + O((\varepsilon - 1)^3).$$

The same result is obtained by an analytic continuation technique. [See also Romeo, paper in preparation.]

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- We began by recalling old results for the surface energy divergences.
- We discussed δ-function potentials, which in general give divergent results in 3rd order, but are finite in strong coupling.
- These divergences are largely identified with surface energies, which can be interpreted as bulk energies when the boundaries are smoothed.
- Precisely analogous phenomena happen for dielectric balls and cylinders (although there is some remarkable symmetry buried in the latter).