Torque on a wedge and an annular piston. II. Electromagnetic Case

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Outline

The electromagnetic Green's dyadic, which corresponds to the vacuum expectation value of the time-ordered product of electric fields, satisfies the differential equation

$$\left(\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\mathbf{1}\right)\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}';\omega)=\mathbf{1}\delta(\mathbf{r}-\mathbf{r}'),$$

or, for the divergenceless dyadic $\boldsymbol{\Gamma}'=\boldsymbol{\Gamma}-\mathbf{1}$,

$$\left(\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\mathbf{1}\right)\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega)=\frac{1}{\omega^2}(\boldsymbol{\nabla}\boldsymbol{\nabla}-\mathbf{1}\nabla^2)\delta(\mathbf{r}-\mathbf{r}'),$$

For a situation with cylindrical symmetry, and perfect conducting boundary conditions, the modes decouple into transverse electric and transverse magnetic modes, and we can write

$$\boldsymbol{\Gamma}'=\mathbf{E}G^E+\mathbf{H}G^H,$$

in terms of transverse electric and magnetic Green's functions, where the polarization tensors have the structure (translationally invariant in z)

$$\mathbf{E}=-\nabla^2(\hat{\mathbf{z}}\times\boldsymbol{\nabla})(\hat{\mathbf{z}}\times\boldsymbol{\nabla}'),$$
$$\mathbf{H}=(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}}))(\boldsymbol{\nabla}'\times(\boldsymbol{\nabla}'\times\hat{\mathbf{z}})).$$

Acting on a completely translationally invariant function,

$$\mathbf{E} + \mathbf{H} = -\nabla_\perp^2(\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\nabla^2),$$

where

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}.$$

Further useful properties of **E** and **H** are

$$\mathrm{Tr}\,\mathbf{E} = \mathrm{Tr}\,\mathbf{H} = \nabla^2\nabla_\perp^2,$$

$$\boldsymbol{\nabla} \times \mathbf{E} \times \overleftarrow{\boldsymbol{\nabla}} = \mathbf{H}\nabla^2, \quad \boldsymbol{\nabla} \times \mathbf{H} \times \overleftarrow{\boldsymbol{\nabla}} = \mathbf{E}\nabla^2,$$

$$\begin{aligned}
\mathbf{E}(\mathbf{r},\mathbf{r}') \cdot \mathbf{H}(\mathbf{r}'',\mathbf{r}''') &= \mathbf{H}(\mathbf{r},\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}'',\mathbf{r}''') = 0,\\
\mathbf{E}(\mathbf{r},\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}'',\mathbf{r}''') &= \mathbf{E}(\mathbf{r},\mathbf{r}''')\nabla_\perp'^2\nabla''^2,\\
\mathbf{H}(\mathbf{r},\mathbf{r}') \cdot \mathbf{H}(\mathbf{r}'',\mathbf{r}''') &= \mathbf{H}(\mathbf{r},\mathbf{r}''')\nabla_\perp'^2\nabla''^2,
\end{aligned}$$

identifying the intermediate coordinates $\mathbf{r}'$ and $\mathbf{r}''$.

Energy

For electomagnetism, the energy density is

$$u = T^{00} = \frac{E^2 + B^2}{2},$$

so by use of the Maxwell equations this becomes, in terms of the imaginary frequency $\zeta = -i\omega$,

$$u = \frac{1}{2}\,\mathrm{Tr}\left[\mathbf{1} + \frac{1}{\zeta^2}\left(\nabla^2\mathbf{1} - \boldsymbol{\nabla}\boldsymbol{\nabla}\right)\right]\cdot\mathbf{EE}.$$

Quantum mechanically, we replace the product of electric fields by the Green's dyadic:

$$\langle\mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \frac{1}{i}\Gamma(\mathbf{r},\mathbf{r}').$$

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$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \frac{1}{i}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}').$$

Because we will be regulating all integrals by point splitting, we can ignore delta functions (contact terms) in evaluations, so since $\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}' = 0$, the quantum vacuum energy is

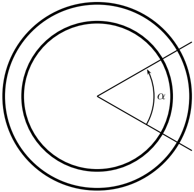
$$
\begin{aligned}
E &= \int (d\mathbf{r}) \langle u(\mathbf{r}) \rangle \\
&= \frac{1}{2i} \int (d\mathbf{r}) \, \mathrm{Tr} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \frac{1}{\zeta^2} (\nabla^2 + \zeta^2) \boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}') \big|_{\mathbf{r}' \to \mathbf{r}} \\
&= \int (d\mathbf{r}) \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} \, \mathrm{Tr} \, \boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}),
\end{aligned}
$$

where in the last equation we have performed the rotation to Euclidean space, so τ is a Euclidean time-splitting parameter, going to zero through positive values. This is a well-known formula. In view of the traces, in terms of the scalar TE and TM Green's functions,

$$
E = -\int (d\mathbf{r}) \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} \zeta^2 (G^E + G^H).
$$

Annular Region

We now specialize to the situation at hand, an annular region bounded by two concentric cylinders, intercut by a co-axial wedge, as illustrated. The inner cylinder has radius a, the outer b, and the wedge angle is α. The axial direction is chosen to coincide with the z axis.



The explicit form for the Green's dyadic is

$$\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}') = -\frac{2}{\alpha}\sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')}\frac{1}{\kappa^2}$$
$$\times\left[\mathbf{E}(\mathbf{r},\mathbf{r}')\cos\nu\theta\cos\nu\theta' g_\nu^E(\rho,\rho') + \mathbf{H}(\mathbf{r},\mathbf{r}')\sin\nu\theta\sin\nu\theta' g_\nu^H(\rho,\rho')\right]$$

Here $\nu = mp$ where $p = \pi/\alpha$. The H mode vanishes on the radial planes, and correspondingly,

$$g_\nu^H(a,\rho') = g_\nu^H(b,\rho') = 0.$$

The normal deriviative of the E mode vanishes on the radial planes, as it does on the circular arcs:

$$\left.\frac{\partial}{\partial\rho} g_\nu^E(\rho,\rho')\right|_{\rho=a,b} = 0.$$

Thus TE modes correspond to scalar Neumann modes, TM, Dirichlet.

TE and TM modes

Both scalar Green's functions satisfy the same equation:

$$\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\kappa^2+\frac{\nu^2}{\rho^2}\right)g_\nu^{E,H}=\frac{1}{\rho}\delta(\rho-\rho').$$

$$\begin{aligned}
g_\nu^H(\rho,\rho') &= I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>) \\
&\quad -\frac{K_\nu(\kappa a)K_\nu(\kappa b)}{\Delta}I_\nu(\kappa\rho)I_\nu(\kappa\rho')-\frac{I_\nu(\kappa a)I_\nu(\kappa b)}{\Delta}K_\nu(\kappa\rho)K_\nu(\kappa\rho') \\
&\quad +\frac{I_\nu(\kappa a)K_\nu(\kappa b)}{\Delta}\left(I_\nu(\kappa\rho)K_\nu(\kappa\rho')+K_\nu(\kappa\rho)I_\nu(\kappa\rho')\right), \\
g_\nu^E(\rho,\rho') &= I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>) \\
&\quad -\frac{K_\nu'(\kappa a)K_\nu'(\kappa b)}{\tilde\Delta}I_\nu(\kappa\rho)I_\nu(\kappa\rho')-\frac{I_\nu'(\kappa a)I_\nu'(\kappa b)}{\tilde\Delta}K_\nu(\kappa\rho)K_\nu(\kappa\rho') \\
&\quad +\frac{I_\nu'(\kappa a)K_\nu'(\kappa b)}{\tilde\Delta}\left(I_\nu(\kappa\rho)K_\nu(\kappa\rho')+K_\nu(\kappa\rho)I_\nu(\kappa\rho')\right),
\end{aligned}$$

where the characteristic denominators are

$$\Delta_\nu(\kappa a, \kappa b) = I_\nu(\kappa b)K_\nu(\kappa a) - I_\nu(\kappa a)K_\nu(\kappa b),$$
$$\tilde{\Delta}_\nu(\kappa a, \kappa b) = I'_\nu(\kappa b)K'_\nu(\kappa a) - I'_\nu(\kappa a)K'_\nu(\kappa b).$$

Point Splitting

Now we have for the energy per length in the z direction

$$\mathcal{E} = -\int \frac{d\zeta}{2\pi}\frac{dk}{2\pi}\zeta^2 e^{i\zeta\tau}e^{ikZ}\int_a^b d\rho\,\rho[g_\nu^E(\rho,\rho)+g_\nu^H(\rho,\rho)],$$

where we have kept the time-difference, and z-difference, nonzero:

$$\tau = t_E - t_E', \quad Z = z - z', \quad \tau, Z \to 0.$$

In the both the Dirichlet and Neumann cases:

$$\int_a^b d\rho\,\rho\,g_\nu^H(\rho,\rho) = \frac{1}{2\kappa}\frac{\partial}{\partial\kappa}\ln\Delta, \quad \int_a^b d\rho\,\rho\,g_\nu^E(\rho,\rho) = \frac{1}{2\kappa}\frac{\partial}{\partial\kappa}\ln\kappa^2\tilde{\Delta},$$

Therefore, the energy per unit length is given by

$$\mathcal{E} = -\frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa^2 f(\kappa\delta,\phi)\sum_m \frac{\partial}{\partial\kappa}\ln\kappa^2\Delta\tilde{\Delta}.$$

Here, to explore the effects of different point-splitting schemes, we write

$$\zeta = \kappa \cos\gamma, \quad k = \kappa \sin\gamma, \quad \tau = \delta \cos\phi, \quad Z = \delta \cos\phi,$$

and then we define the cutoff function

$$f(\kappa\delta, \phi) = \int_0^{2\pi} \frac{d\gamma}{2\pi} \cos^2\gamma e^{i\kappa\delta\cos(\gamma-\phi)},$$

which equals $1/2$ for $\delta = 0$. For finite δ, temporal splitting corresponds to

$$f(\kappa\delta, 0) = J_0(\kappa\delta) - \frac{1}{\kappa\delta} J_1(\kappa\delta),$$

while z-splitting corresponds to

$$f(\kappa\delta, \pi/2) = \frac{1}{\kappa\delta} J_1(\kappa\delta).$$

Torque

To compute the torque on one of the radial planes, we need to compute the angular component of the stress tensor,

$$\langle T^{\theta\theta}\rangle = -\frac{1}{2}\langle E_\theta^2 - B_\rho^2 - B_z^2\rangle$$
$$= -\frac{1}{2i}\left[\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\Gamma}\cdot\hat{\boldsymbol{\theta}} + \frac{1}{\omega^2}\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\Gamma}\times\overleftarrow{\boldsymbol{\nabla}}'\cdot\hat{\boldsymbol{\rho}} + \frac{1}{\omega^2}\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\Gamma}\times\overleftarrow{\boldsymbol{\nabla}}'\cdot\hat{\mathbf{z}}\right].$$

The torque then is immediately obtained by integrating this over one radial side of the annular region:

$$\tau = \int_a^b d\rho\,\rho\,\langle T^{\theta\theta}\rangle = \frac{1}{\alpha}\sum_m \nu^2\int_0^\infty \frac{d\kappa\,\kappa}{4\pi}J_0(\kappa\delta)\int_a^b\frac{d\rho}{\rho}[g_\nu^E(\rho,\rho)+g_\nu^H(\rho,\rho)].$$

Similar results are obtained for the radial integral for the TM and TE parts:

$$\int_a^b\frac{d\rho}{\rho}g_\nu^H(\rho,\rho) = -\frac{\alpha}{2\nu^2}\frac{\partial}{\partial\alpha}\ln\Delta,\quad \int_a^b\frac{d\rho}{\rho}g_\nu^E(\rho,\rho) = -\frac{\alpha}{2\nu^2}\frac{\partial}{\partial\alpha}\ln\tilde{\Delta}.$$

Neutral direction?

Thus the electromagnetic torque on one of the planes is

$$\tau = -\frac{\partial}{\partial\alpha}\frac{1}{4\pi}\sum_m \int_0^\infty d\kappa\kappa\, J_0(\kappa\delta)\ln\Delta\tilde{\Delta}.$$

Comparing with the expression for the energy, we see this indeed the negative derivative with respect to the wedge angle of the interior energy provided $\phi = \pi/2$, that is, for point-splitting in the z direction. We will now proceed to evaluate the energy, by expliciting isolating the divergent contributions as $\delta \to 0$, and extract the finite parts. Will it be true, as in the scalar case, that after renomalization the finite torque is equal to the negative derivative of the finite energy with respect to the wedge angle?

Divergent terms for TE energy

We now turn to the examination of the Neumann or TE contribution to the Casimir energy of the annular region. That energy is

$$\tilde{\mathcal{E}} = -\frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa^2 f(\kappa\delta,\phi)\sum_{m=0}^\infty \frac{\partial}{\partial\kappa}\ln\kappa^2\tilde{\Delta}.$$

As in the Dirichlet case, we expand the Bessel functions according to the uniform asymptotic expansion, which here reads

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu t}}\frac{1}{z}e^{\eta\nu}\left(1+\sum_{k=1}^\infty \frac{v_k(t)}{\nu^k}\right),$$

$$K'_\nu(\nu z) \sim -\sqrt{\frac{\pi}{2\nu t}}\frac{1}{z}e^{-\eta\nu}\left(1+\sum_{k=1}^\infty (-1)^k\frac{v_k(t)}{\nu^k}\right),$$

where $t = (1+z^2)^{-1/2}$, $d\eta/dz = 1/(zt)$, and the $v_k(t)$ are polynomials of degree $3k$.

Asymptotic behavior of integrand:

Because of this behavior, the second product of Bessel functions in $\tilde{\Delta}$ is exponentially subdominant. Thus the logarithm in the energy is

$$\ln \kappa^2 \tilde{\Delta} \sim \text{constant} + \nu[\eta(z) - \eta(\tilde{z})] + \left(t^{-1/2} + \tilde{t}^{-1/2}\right) + \ln\left(1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k}\right) + \ln\left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(\tilde{t})}{\nu^k}\right),$$

where $\tilde{z} = za/b$, $\tilde{t} = (1 + \tilde{z}^2)^{-1/2}$. Here the constant means a term independent of κ, which will not survive differentiation. Note that the $1/z$ behavior seen in the prefactors in the UAE is cancelled by the multiplication of $\tilde{\Delta}$ by κ^2. In the following, we will consider the z-splitting regulator, $\phi = \pi/2$, since the result for time-splitting may be obtained by differentiation:

$$\tilde{\mathcal{E}}(0) = \frac{\partial}{\partial \delta}[\delta \tilde{\mathcal{E}}(\pi/2)].$$

Leading Weyl divergence

We now extract the divergences, that is, the terms proportional to nonpositive values of δ, just as in I. We label those terms by the corresponding power of $1/\delta$. The calculation closely parallels that in I, except for the additional zero mode, $m = 0$. Except for that term, the leading divergence is exactly that found in I,

$$\tilde{\mathcal{E}}_4^{m>0} = -\frac{\alpha(b^2 - a^2)}{4\pi^2\delta^4} + \frac{b-a}{8\pi\delta^3}.$$

However, the $m = 0$ term yields

$$\tilde{\mathcal{E}}_4^{m=0} = -\frac{b-a}{4\pi\delta^4},$$

thereby (correctly) reversing the sign of the second term in Eq. (7). Thus the leading divergence is again the expected Weyl volume divergence:

$$\tilde{\mathcal{E}}^{(4)} = -\frac{A}{2\pi^2\delta^4}, \quad A = \frac{1}{2}\alpha(b^2 - a^2).$$

Perimeter and corner divergences

Evidently, the $O(\nu^{-3})$ term, for $m > 0$, is exactly reversed in sign from that for the Dirichlet term,

$$\tilde{\mathcal{E}}_3^{m>0} = -\frac{\alpha(a+b)}{16\pi\delta^3} + \frac{1}{8\pi\delta^2},$$

but again the sign of the subleading term is reversed by including $m = 0$:

$$\tilde{\mathcal{E}}_3^{m=0} = -\frac{1}{4\pi\delta^2}.$$

Thus, we get the correct surface area and corner terms:

$$\tilde{\mathcal{E}}^{(3)} = -\frac{P}{16\pi\delta^3}, \quad P = \alpha(a+b) + 2(b-a),$$
$$\tilde{\mathcal{E}}^{(2)} = -\frac{C}{48\pi\delta^2}, \quad C = 6.$$

Subleading divergences

Closely following the path blazed in computing the divergent terms coming from the polynomial asymptotic corrections in the Dirichlet case in I, but including the $m = 0$ terms, we find

$$\tilde{\mathcal{E}}_2 = \frac{3}{64\pi}\frac{1}{\delta}\left(\frac{1}{a} - \frac{1}{b}\right),$$
$$\tilde{\mathcal{E}}_1 = -\frac{5}{1024}\frac{\alpha}{\pi}\frac{1}{\delta}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{3\ln\delta}{128\pi}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

$m = 0$ terms

Before proceeding, it is time to recognize that use of the uniform asymptotic expansion is apparently inconsistent for $m = 0$, because $\nu = 0$ then. So let us calculate the $m = 0$ contribution directly from

$$\tilde{\mathcal{E}}_{m=0} = -\frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa^2 \frac{J_1(\kappa\delta)}{\kappa\delta}\frac{\partial}{\partial\kappa}\ln\kappa^2[I_0'(\kappa b)K_0'(\kappa a) - I_0'(\kappa a)K_0'(\kappa b)],$$

where the divergent terms arise from the large argument expansions

$$I_0'(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{3}{8z} - \frac{15}{128z^2} + \dots\right),$$

$$K_0'(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}\left(1 + \frac{3}{8z} - \frac{15}{128z^2} + \dots\right),$$

Inserting this into $\tilde{\mathcal{E}}_{m=0}$ we obtain

$$\tilde{\mathcal{E}}_{m=0} \sim -\frac{1}{4\pi\delta}\int_0^\infty d\kappa\, J_1(\kappa\delta)\left[\kappa(b-a) + 1 + \frac{3}{8\kappa}\left(\frac{1}{b} - \frac{1}{a}\right) + \frac{3}{8\kappa^2}\left(\frac{1}{b^2} + \frac{1}{a\right.$$

$$\sim -\frac{b-a}{4\pi\delta^3} - \frac{1}{4\pi\delta^2} + \frac{3}{32\pi\delta}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{3}{64\pi}\ln\mu\delta\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

IR divergence; remaining log divergence

Here, in the last term we introduced a mass, $\kappa^2 \to \kappa^2 + \mu^2$, in order to eliminate the infrared divergence. These terms all agree with the corresponding terms found from the uniform asysmptotic expansion by taking $m = 0$.

There is one remaining divergent term, arising from the $1/\nu^3$ term, but here we exclude $m = 0$, because that subtraction is not necessary to make the $m = 0$ contribution to the energy finite at $\delta = 0$. That term is

$$\tilde{\mathcal{E}}_0 \sim \frac{\alpha}{180\pi^2} \ln \delta \left(\frac{1}{b^2} - \frac{1}{a^2} \right).$$

Summary of divergences

Let us summarize the divergent terms for the Neumann or TE modes:

$$
\begin{aligned}
\tilde{\mathcal{E}}_{\rm div} = & -\frac{A}{2\pi^2\delta^4} - \frac{P}{16\pi\delta^3} - \frac{C}{48\pi\delta^2} \\
& + \frac{3}{64\pi\delta}\left(\frac{1}{a} - \frac{1}{b}\right) - \frac{5\alpha}{1024\pi\delta}\left(\frac{1}{a} + \frac{1}{b}\right) \\
& + \frac{3\ln\delta}{128\pi}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{\alpha\ln\delta}{180\pi^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right).
\end{aligned}
$$

This small-δ Laurent expansion exactly agrees with that found by the heat-kernel calculation of Nesterenko, Pirozhenko, and Dittrich, who consider a wedge intercut with with a single coaxial circular cylinder with radius R.

Relation between heat kernel and cylinder kernel

From the latter heat-kernel coefficients the cylinder-kernel coeffiencts can be readily extracted. The cylinder kernel $T(t)$ is defined in terms of the eigenvalues of the Laplacian in d dimensions,

$$T(t) = \sum_j e^{-\lambda_j t} \sim \sum_{s=0}^{\infty} e_s t^{s-d} + \sum_{\substack{s=d+1 \\ s-d \text{odd}}} f_s t^{s-d} \ln t,$$

where the expansion holds as $t \to 0$ through positive values. The energy is given by

$$E(t) = -\frac{1}{2}\frac{\partial}{\partial t} T(t),$$

which corresponds to the energy computed here with $\phi = 0$, that is, time-splitting. In view of the relation between z and t splitting, we see that the z-splitting result should be identical to that of $-\frac{1}{2t} T(t)$ with $t \to \delta$.

In this way we transcribe the results of Nesterenko et al., for the cylinder kernel per unit length:

$$-\frac{1}{2t}T(t) \sim -\frac{A}{2\pi^2 t^4} - \frac{P}{16\pi t^3} - \frac{1}{16\pi^2 t^2} + \frac{3 - 5\alpha/16}{64\pi R t} + \frac{\ln t}{16\pi^2 R^2}\left(\frac{3\pi}{8} - \frac{4\alpha}{45}\right).$$

This exactly agrees with our result when $a \to R$ and $b \to \infty$ (except in the first two terms). The reason for the factor of 2 discrepancy in the third (corner) term is that Nesterenko et al. have only two corners, not four.

Extraction of finite part

Just as in the Dirichlet case considered in I, the divergent terms have finite remainders, which we state here (the $m = 0$ terms do not contribute to these):

$$\begin{aligned}\tilde{\mathcal{E}}_f &= -\frac{\pi^2}{2880\alpha^3}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)-\frac{\zeta(3)}{64\pi\alpha^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{576\alpha}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\\ &+\frac{1}{64\pi b^2}\left[\frac{13}{8}+\gamma-\ln 4b\alpha+3\ln\mu\right]+(b\to a)\\ &+\frac{\alpha}{\pi b^2}\left(-\frac{1}{180\pi}\ln\frac{b\alpha}{\pi}+\frac{1079}{69120}\right)-(b\to a)\\ &+\frac{29}{46080}\frac{\alpha^2}{\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\frac{5}{12012}\frac{\alpha^3}{\pi^4}\zeta(3)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\tilde{\mathcal{E}}_R.\end{aligned}$$

The last two explicitly given terms are what come from the next two terms in the uniform expansion for $m > 0$. Note that we have made no approximation here, we have merely added and subtracted the leading terms in the uniform asymptotic expansion of the integrand for the energy.

Remainder

The remainder, therefore, consists of two parts, that arising from $m = 0$:

$$\tilde{\mathcal{E}}_{R0} = -\frac{1}{8\pi}\int_0^\infty d\kappa\,\kappa\left[\kappa\frac{\partial}{\partial\kappa}\ln\kappa^2\tilde{\Delta}_{m=0} - \kappa(b-a) - 1 - \frac{3}{8\kappa}\left(\frac{1}{b}-\frac{1}{a}\right) - \frac{3}{8(\kappa^2+\mu^2)}\left(\frac{1}{b^2}+\frac{1}{a^2}\right)\right],$$

and the rest coming from the terms with $m > 0$:

$$\tilde{\mathcal{E}}_R' = -\frac{1}{8\pi b^2}\sum_{m=1}^\infty \nu^3\int_0^\infty dz\,z^2\left[\tilde{f}(\nu,z,a/b) + \sum_{n=4}^{-2}\tilde{f}_m(\nu,z,a/b)\right].$$

Here, with the abbreviations $I = I_\nu(\nu z)$, $\tilde{I} = I_\nu(\nu z a/b)$, etc., the log term is

$$\tilde{f} = \frac{\left(1+\frac{1}{z^2}\right)(I\tilde{K}' - K\tilde{I}') + \frac{a}{b}\left(1+\frac{b^2}{a^2z^2}\right)(I'\tilde{K} - K'\tilde{I})}{I'\tilde{K}' - K'\tilde{I}'}.$$

Subtractions

The subtractions are easily read off:

$$\tilde{f}_4 = -\frac{1}{zt} + \frac{a}{b}\frac{1}{\tilde{z}\tilde{t}}, \quad \tilde{f}_3 = -\frac{1}{2\nu}(zt^2 + \frac{a}{b}\tilde{z}\tilde{t}^2),$$

$$\tilde{f}_2 = \frac{1}{8\nu^2} zt^3(-3 + 7t^2) - \frac{a}{b}(z \to \tilde{z}),$$

$$\tilde{f}_1 = \frac{1}{8\nu^3} zt^4(-3 + 20t^2 - 21t^4) + \frac{a}{b}(z \to \tilde{z}),$$

$$\tilde{f}_0 = \frac{1}{5760\nu^4} zt^5(-2835 + 39105t^2 - 99225t^4 + 65835t^6) - \frac{a}{b}(z \to \tilde{z}),$$

$$\tilde{f}_{-1} = \frac{1}{128\nu^5} zt^6(-108 + 2616t^2 - 11728t^4 + 17640t^6 - 8484t^8) + \frac{a}{b}(z \to \tilde{z}),$$

$$\tilde{f}_{-2} = \frac{1}{32560\nu^6} zt^7(-598185 + 22680945t^2 - 156073050t^4 + 393353730t^6 - 415212525t^8 + 156010365t^{10}) - \frac{a}{b}(z \to \tilde{z}).$$

Numerics

Recall how this worked in the Dirichlet case. There the finite part (corrected) is

$$\mathcal{E}_4^f = -\frac{\pi^2}{2880}\frac{1}{\alpha^3}\left(\frac{1}{a^2}-\frac{1}{b^2}\right),$$

$$\mathcal{E}_3^f = \frac{\zeta(3)}{64\pi}\frac{1}{\alpha^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

$$\mathcal{E}_2^f = -\frac{1}{144}\frac{1}{\alpha}\left(\frac{1}{a^2}-\frac{1}{b^2}\right),$$

$$\mathcal{E}_1^f = \frac{1}{128\pi}(\gamma+\frac{7}{4}-\ln 4b\alpha/\mu)\frac{1}{b^2}+(b\to a),$$

$$\mathcal{E}_0^f = \frac{\alpha}{4\pi^2}\left(\frac{1}{315}\ln\frac{\pi\mu}{b\alpha}-\frac{397}{15120}\right)\frac{1}{b^2}-(b\to a).$$

$$\mathcal{E}_{-1} = -\frac{\alpha^2}{6144\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

$$\mathcal{E}_{-2} = -\frac{29}{180180}\frac{\zeta(3)}{\pi^4}\alpha^3\left(\frac{1}{a^2}-\frac{1}{b^2}\right).$$

Energy for Dirichlet contribution

The total finite energy is the sum of these finite terms plus the remainder:

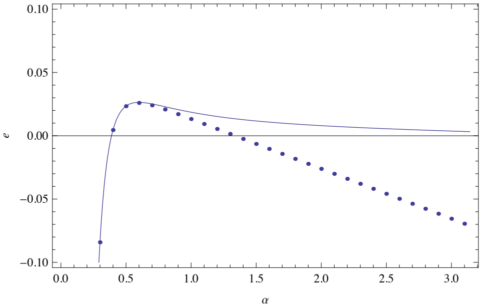
$$\mathcal{E}^f = \sum_{n=4}^{-2} \mathcal{E}_n^f + \mathcal{E}_R,$$

where the remainder has a similar form to that for the Neumann case:

$$\mathcal{E}_R = -\frac{1}{8\pi b^2} \sum_{m=1}^{\infty} \nu^3 \int_0^\infty dz\, z^2 \left[f(\nu, z, a/b) + \sum_{n=4}^{-2} f_m(\nu, z, a/b) \right],$$

where the expressions for the original integrand and the subtractions are given in I. The following figure gives a typical result of the calculation (corrected).

Numerical results, Dirichlet



Energy of a finite annular region, with $a/b = 0.5$. The solid curve is the sum of the explicit finite contributions, while the dotted points are the total energy including the remainder.

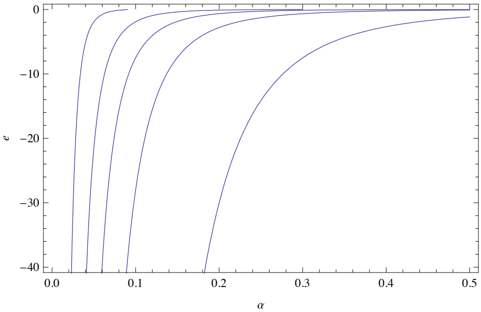
Renormalization

The total energy becomes a linear function of α for sufficiently large wedge angles. Because of the logarithmic terms in the divergent parts in the energy, the linear terms are undetermined. That is, we can add to the energy an arbitrary term of the form

$$\mathcal{E}_{ct} = A + B\alpha.$$

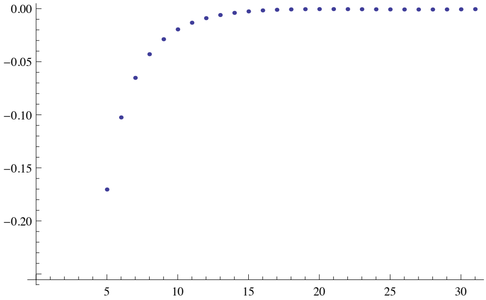
We subtract off the linear behavior seen in the previous figure, because the energy should approach zero for sufficiently (but not very) large α. In this way, we get the energies seen in the following figure.

Renormalized results, Dirichlet



Renormalized energies for $a/b = .9, .7, .5, .3, .1$.

Similar results hold for the Neumann (TE) case:



Renormalized energy for TE modes for $a/b = 0.5$.

Conclusions

Because of curvature divergences, it is impossible to extract a unique finite part of the energy. However, the divergences are all constant or linear in the wedge angle α. Therefore, we can renormalize the energy by subtracting the linear dependence for large angles, to make the energy go to zero when the separation between the wedge planes is large. The resulting energy is completely finite, independent of regularization scheme, and exhibits no torque anomaly:

$$\tau = -\frac{\partial}{\partial \alpha}\mathcal{E}(\alpha).$$

These results, of course, are consistent with, and generalize to electromagnetism, the annular piston work with Jef Wagner and Klaus Kirsten.