The local torque calculation II

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Where We Left Off

The asymptotic expansion of the second term in $\bar{T}$ arising from the radial Green's function takes the form

$$\bar{T}_2 \sim \frac{2}{\pi\alpha}\mathcal{A}\int_0^\infty \kappa\, d\kappa \frac{J_0(\kappa\delta)e^{2\nu(\eta_a-\eta_\rho)}}{\nu\sqrt{1+z(\rho)^2}}[1+\cdots] \tag{25}$$

where

$$\mathcal{A} = \sum_{m=1}^{\infty}\sin(\nu\theta)\sin(\nu\theta') \qquad \nu = \frac{m\pi}{\alpha}$$

$$\kappa^2 = \omega^2 + k^2.$$

Review of $\bar{T}$

Stepping back several steps, we started with

$$\bar{T} = -\frac{2}{\pi^2 \alpha} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \, e^{i\omega t + ikz} \mathcal{A} \, g_\nu(\rho, \rho') \tag{12}$$

where

$$g_\nu(\rho, \rho') = I_\nu(\kappa\rho_<) K_\nu(\kappa\rho_>) - K_\nu(\kappa\rho) K_\nu(\kappa\rho') \frac{I_\nu(\kappa a)}{K_\nu(\kappa a)}$$

and $\mathcal{A}$, κ, and ν are as defined before.

Review of $\bar{T}$

Transforming to polar coordinates in the ω-k and t-z planes, we define

$$\omega = \kappa \cos(\gamma) \qquad t = \delta \cos(\phi)$$

$$k = \kappa \sin(\gamma) \qquad z = \delta \sin(\phi),$$

and obtain

$$\bar{T} = -\frac{2}{\pi^2 \alpha} \int_0^\infty d\kappa\, \kappa \int_0^{2\pi} d\gamma\, e^{i\kappa\delta \cos(\gamma - \phi)} \sum_{m=1}^{\infty} \sin(\nu\theta) \sin(\nu\theta')\, g_\nu(\rho, \rho').$$

Review of $\bar{T}$

If we write the sine functions in terms of exponentials, then (almost) everything within the m summation becomes a product of exponentials, so

$$\begin{aligned} \sin(\nu\theta)\sin(\nu\theta') &= -\frac{1}{4}(e^{i\nu\theta} - e^{-i\nu\theta})(e^{i\nu\theta'} - e^{-i\nu\theta'}) \\ &= -\frac{1}{4}\left(e^{i\nu(\theta+\theta')} - e^{i\nu(\theta-\theta')} - e^{-i\nu(\theta-\theta')} + e^{-i\nu(\theta+\theta')}\right). \end{aligned}$$

By the symmetry of this expression, we can look at just the first term and add in the rest at the end.

$$\bar{T}' = -\frac{1}{2\pi^2\alpha}\int_0^\infty d\kappa\,\kappa\int_0^{2\pi} d\gamma \sum_{m=1}^\infty e^{i\kappa\delta\cos(\gamma-\phi)+i\nu(\theta+\theta')}\,g_\nu(\rho,\rho')$$

Uniform Asymptotic Expansion

Taking the uniform asymptotic expansions of the modified bessel functions yields exponential factors as well, namely with

$$I_\nu(\nu z) \sim \sqrt{\frac{t}{2\pi\nu}} e^{\nu\eta} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right)$$

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi t}{2\nu}} e^{-\nu\eta} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{\nu^k}\right)$$

we find

$$\bar{T}_2' \sim \frac{1}{4\pi^2\alpha} \int_0^\infty d\kappa\, \kappa \int_0^{2\pi} d\gamma \sum_{m=1}^{\infty} \frac{1}{\nu\sqrt{1+z(\rho)^2}} \times e^{2\nu(\eta_a-\eta_\rho)+i\kappa\delta\cos(\gamma-\phi)+i\nu(\theta+\theta')} \left(1 - \frac{2u_1(t)}{\nu} + \cdots\right).$$

Change of Variables

Next we make a change of variables and pull any extraneous factors outside of the summation. We take

$$\kappa\rho = z\nu, \quad d\kappa\,\rho = dz\,\nu$$

and so

$$\begin{aligned}
\bar{T}_2' \sim & \frac{1}{4\pi^2\alpha\rho^2}\int_0^\infty \frac{dz\,z}{\sqrt{1+z^2}}\int_0^{2\pi} d\gamma \\
& \sum_{m=1}^\infty \nu\, e^{\nu\left(2(\eta_a-\eta_\rho)+i\frac{z\delta}{\rho}\cos(\gamma-\phi)+i(\theta+\theta')\right)}\left(1-\frac{2u_1(t)}{\nu}+\cdots\right) \\
\sim & \frac{1}{4\pi\alpha^2\rho^2}\int_0^\infty \frac{dz\,z}{\sqrt{1+z^2}}\int_0^{2\pi} d\gamma \\
& \sum_{m=1}^\infty m\left(e^{\frac{2\pi}{\alpha}(\eta_a-\eta_\rho)+\frac{i\pi z\delta}{\rho\alpha}\cos(\gamma-\phi)+\frac{i\pi}{\alpha}(\theta+\theta')}\right)^m\left(1-\frac{\alpha}{\pi}\frac{2u_1(t)}{m}+\cdots\right).
\end{aligned}$$

Polylogarithms

For ease of notation, we define a new function

$$f^{++}(z) = \frac{2\pi}{\alpha}(\eta_a - \eta_\rho) + \frac{i\pi z\delta}{\rho\alpha}\cos(\gamma - \phi) + \frac{i\pi}{\alpha}(\theta + \theta').$$

We also note that sums involving inverse powers of the summation index are called *polylogarithms* and have the definition

$$\mathrm{Li}_s(z) \stackrel{def}{=} \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

These simplify our earlier expression to

$$\bar{T}_2' \sim \frac{1}{4\pi\alpha^2\rho^2} \int_0^\infty \frac{dz\, z}{\sqrt{1+z^2}} \int_0^{2\pi} d\gamma \left[\mathrm{Li}_{-1}(e^{f^{++}(z)}) - \frac{2\alpha}{\pi}\mathrm{Li}_0(e^{f^{++}(z)})u_1(t) + \cdots\right]$$

Polylogarithms

Some are known in closed form,

$$\mathrm{Li}_{-1}(e^u) = \frac{e^u}{(1-e^u)^2}$$
$$\mathrm{Li}_0(e^u) = \frac{e^u}{1-e^u}$$
$$\mathrm{Li}_1(e^u) = -\ln(1-z),$$

so

$$\bar{T}_2' \sim \frac{1}{4\pi\alpha^2\rho^2}\int_0^\infty \frac{dz\, z}{\sqrt{1+z^2}}\int_0^{2\pi} d\gamma \left[\frac{e^{f^{++}(z)}}{(1-e^{f^{++}(z)})^2} - \frac{2\alpha}{\pi}\frac{e^{f^{++}(z)}}{1-e^{f^{++}(z)}}u_1(t) + \cdots\right].$$

What's Next

- Find the divergent and finite parts of $\bar{T}_2$.
- Take second partial derivatives of $\bar{T}$ and evaluate similarly to find components of the stress tensor.
- Construct $\bar{T}$ similarly near corners.