NETS AND LATTICES IN EUCLIDEAN SPACES AND COCYCLES OVER THE WEYL CHAMBER FLOW

ANATOLE KATOK

A subset $N \subset \mathbb{R}^k$ is a *separated net*, if for some positive numbers $r < R$ every ball of radius r contains at most one point form N and every ball of radius R contains at least one point from N. Let α be a Borel (or measure-preserving) action of $\mathbb{R}^k$ on a Borel (or Lebesgue) space X, A Borel set $S \subset X$ is called a *uniform section* if the intersection of S with any α orbit is a separated net. See [3] for the corresponding definition in the measure-preserving case.

The question of whether any separated net is bi-Lipschits equivalent to the lattice $\mathbb{Z}^k$ is a particular case of a general question posed by Gromov. It was answered in the negative in 1998 independently by Burago and Kleiner [1] and by McMullen [4]. The corresponding question for the $\mathbb{R}^k$ actions is whether a uniform section possesses a Borel (or measurable) $\mathbb{Z}^k$ structure bi-Lipshtitz along the orbits. Apparently this question was raised by Furstenberg in the sixties long before Gromov. Notice that in this case even existence of a section with such a $\mathbb{Z}^k$ structure is not immediately apparent if $k \geq 2$, and was proved in [3].

Burago-Kleiner construction can be adapted to provide a counter-example to the Furstenberg question; however, in such an adaptation the action is specially constructed simultaneously with the section. In other words, such examples are not "natural".

We provide a class of natural counter-examples to the Furstenberg question. Unlike the Burago-Kleiner and McMullen methods which are of analytic nature our method is based on geometry and ergodic theory of simple Lie groups. We restrict ourselves to the most basic case where the Lie group in question is $SL(n, \mathbb{R})$.

Let $n \geq 3$ and let Γ be a co-compact lattice in $SL(n, \mathbb{R})$. Recall that *Weyl chamber flow* is the action of the subgroup D_+ of positive diagonal matrices of the homogeneous space $SL(n, \mathbb{R})/\Gamma$ by left translations. For $1 \leq i \neq j \leq n$ we denote by $\mathcal{U}_{ij}$ the one-dimensional homogeneous foliation into the left cosets of the one-parameter unipotent group $\exp tN_{ij},\ t \in R$, where N_{ij} is the matrix with one at the intersection of the ith row and the jth column and zeroes elsewhere.

Theorem 1. *Let S be a piecewise smooth uniform section of the Weyl chamber flow with the extra property that for some $r > 0$ its intersection with any leaf of every foliation $\mathcal{U}_{ij}$ does not contain two points at the distance less than r in the inner metric of the leaf. Then S does not possess a Lebesgue measurable $\mathbb{Z}^k$ structure bi-Lipshtitz along the orbits of the Weyl chamber flow.*

The main ingredient in the proof is an extension of the cocycle rigidity results for Hölder cocycles over Weyl chamber flows and related actions from [2] to piece-wise smooth cocycles.

References

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: katok_a@math.psu.edu

URL: http://www.math.psu.edu/katok_a