Texas Geometry and Topology Conference

This is a report on the presentations at the 44th meeting of the Texas Geometry and Topology Conference at Texas A&M University, November 12-14, 2010. This conference was partially supported by National Science Foundation Grant DMS-0904481 and Texas A&M University. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

Meeting 44. Texas A&M University, November 12–14, 2010

Gerard Besson, Université de Grenoble, Collapsing irreducible manifolds with nontrivial fundamental group

We present a result which is used in the last part of the proof of the Geometrization of 3-manifolds following Perelman. It concerns the structure of collapsing closed 3-manifolds. Here by collapsing we mean that the closed manifold carries a sequence of Riemannian metrics so that locally the sectional curvature is bounded from below and that the metric balls have small volume. The precise definition comes from Perelman and will be given. We show that they are graph-manifolds, that is, obtained by gluing Seifert manifolds with boundaries along tori. Several alternative proofs have been given and the originality of this one is that it relies on Thurston’s Geometrization of Haken manifolds. Using this powerful tool, the proof that will be presented only uses two covering arguments. One of them is borrowed from M. Gromov (see [Gromov]). The argument essentially shows that if a closed Riemannian manifold with Ricci curvature bounded below is such that all balls of radius 1 have small volume then the manifold has a “low complexity”. A tool for measuring the complexity is Gromov’s simplicial volume. The main references are listed below.

References


Gerard Besson, Université de Grenoble, The Geometrization Program

We shall briefly present the geometrization program. It starts by a quick description of the uniformization of surfaces. From that we proceed to the eight geometries in dimension 3, illustrating them by few examples. The main reference for this part of the lecture is [Bon]. We shall then describe the technique called Ricci flow, first by presenting the main ideas on a simple version concerning curves: the so-called curve shortening process. We then describe the general Ricci flow and, in particular, the main differences with the one-dimensional case: presence of surgeries. A quick description of the surgery process will probably end the lecture. The main reference for this part is [Chow-Knopf].
Robert Bryant, MSRI, The affine Bonnet problem

The classical Euclidean problem studied by Bonnet was to determine whether, and in how many ways, a Riemannian surface can be isometrically embedded into Euclidean 3-space so that its mean curvature is a prescribed function. He found that, generically, specifying the metric and mean curvature allowed no solution but that there are special cases in which, not only are there solutions, but there are even 1-parameter families of distinct solutions. Much later, these ‘Bonnet surfaces’ were found to be intimately connected with integrable systems and Lax pairs.

I will consider the analogous problem in affine geometry: To determine whether, and in how many ways, a surface endowed with a Riemannian metric $g$ and a function $H$ can be immersed into affine 3-space in such a way that the induced Blaschke metric is $g$ and the induced affine mean curvature is $H$. This affine problem is, in many ways, richer and more interesting than the corresponding Euclidean problem. I will classify the pairs $(g, H)$ that display the greatest flexibility in their solution space and say what is known about the (suspected) links with integrable systems and Lax pairs.

Peter Ebenfelt, University of California, San Diego, Rigidity of CR mappings into hyperquadrics

This talk will be devoted to certain rigidity properties of maps into hyperquadrics that arise in CR geometry. Similar rigidity phenomena occur in e.g. Riemannian geometry and complex geometry.

The simplest example of a strictly pseudoconvex hypersurface in complex space is the unit sphere, locally the only such with vanishing CR curvature. It was discovered by Poincaré (see also Alexander [A74]) that a local non-constant holomorphic mapping sending a piece of the unit sphere in $\mathbb{C}^{n+1}$ into itself must in fact be a global holomorphic automorphism of $\mathbb{C}^{p+1}$ preserving the sphere. Much later, Faran [Fa86] and Webster [W79] found that such rigidity persists for local mappings of a piece of the sphere in $\mathbb{C}^{n+1}$ into the sphere in $\mathbb{C}^{N+1}$ provided that $N - n < n$. More precisely, if $N - n < n$ then any such mapping is the standard linear embedding composed with an automorphism of the target sphere. When $N - n = n$ this is no longer true; there is a quadratic map, which was discovered by H. Whitney. These results were the starting point of a currently very active, long-term project of classifying maps into spheres (or more generally hyperquadrics).

About 5 years ago, the speaker jointly with X. Huang and D. Zaitsev [EHZ04], [EHZ05] proved that there is also rigidity for mappings of strictly pseudoconvex hypersurfaces of low CR complexity into spheres. The CR complexity $\mu(M)$ of a strictly pseudoconvex hypersurface $M \subset \mathbb{C}^{n+1}$ is defined to be the minimum of positive integers $N_0 - n$ such that there is a non-constant map $f_0$ sending $M$ into the sphere in $\mathbb{C}^{N_0+1}$. (Thus, $\mu(M) = 0$ if and only if $M$ is spherical.) The main result is that if $f$ is a non-constant map of $M$ into the unit sphere in $\mathbb{C}^{N+1}$ and if $N - n + \mu(M) < n$, then $f = T \circ L \circ f_0$ where $L$ is the linear embedding and $T$ an automorphism of the target sphere. Applications of this result include:

(i) an extension of the Poincaré-Alexander result mentioned above to strictly pseudoconvex domains of low CR complexity

References


(ii) a characterization, up to birational equivalence, of isolated low-codimensional singularities in terms of the local CR geometry of their Milnor links.

More recently, there has been progress on rigidity question related to non-trivial maps of a Levi nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$ of a given, positive signature $\ell$ into the hyperquadric of the same signature $\ell$ in $\mathbb{C}^{N+1}$. Here, a new phenomenon occurs. If the CR complexity (suitably refined) is sufficiently small, then rigidity persists regardless of the codimension $N-n$. Also, there is a partial rigidity for mappings into hyperquadrics of higher signature provided the signature difference is small.

In this talk, I will give the background in the strictly pseudoconvex case and explain the more recent work in the positive signature case.

References


Phillip Griffiths, Institute for Advanced Study, *Hodge theory and representation theory*

This talk is based in part on joint work with Mark Green and Matt Kerr.

This talk will present an informal overview of the possible relationship between the subjects of Hodge theory and representation theory. In Hodge theory the basic objects are polarized Hodge structures of weight $n$. The basic symmetry groups of the theory are the Mumford-Tate groups $M$; they encode both the rational structure and the Hodge decomposition. The smaller $M$ is, the more symmetry the polarized Hodge structure has, with one extreme being when $M$ is an anisotropic $\mathbb{Q}$-algebraic torus, in which case the Hodge structure is of CM (complex multiplication)-type.

In representation theory the basic objects for this talk are the discrete series representations, viewed as the infinite part of cuspidal automorphic representations of $G(\mathbb{A})$, where $G$ is a reductive algebraic group over $\mathbb{Q}$ and $\mathbb{A}$ are the adeles. A recent observation is: Mumford-Tate groups are exactly those for which one has cuspidal automorphic representations.

This suggests a possible connection:

| Hodge theory | arithmetic representation theory |

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In the classical weight \( n = 1 \) case, i.e. the theory of abelian varieties, this connection is well established and is the subject of an extensive and deep theory. In the non-classical case, when automorphic forms on the right are replaced by automorphic cohomology whose arithmetic significance is at best mysterious, the understanding of what the relationship might be is in its very earliest stages. The objective of the talk will be to explain some of the basic issues that are involved.

Jun-Muk Hwang, KIAS, Seoul, Korea, Symmetries of cone structures

A cone structure on a complex manifold \( M \) is a submanifold \( C \subset \mathbb{P}T(M) \) in the projectivized tangent bundle which is surjective over \( M \). A classical example is the (holomorphic) conformal structure where the fiber \( C_x \subset \mathbb{P}T_x(X) \) is a quadric hypersurface for each point \( x \in M \). Various cone structures have been studied in integral geometry and twistor theory since 1970’s (cf.[Ma]). In my joint work with Ngaiming Mok in 1995-2005, many interesting examples of cone structures have been discovered in algebraic geometric setting, arising from minimal rational curves. These cone structures, called VMRT, play an important role in algebraic geometry and many algebraic geometric questions on Fano manifolds can be answered by studying differential geometric properties of these cone structures (cf.[HM99]).

An important problem is to study the infinitesimal symmetries of the cone structure, i.e., the Lie algebra of local holomorphic vector fields on \( M \) preserving the cone structure \( C \subset \mathbb{P}T(M) \). This immediately leads to the study of the prolongations \( \text{aut}(\hat{S})^{(i)} \) of the Lie algebra \( \text{aut}(\hat{S}) \subset \text{gl}_n(\mathbb{C}) \) of the infinitesimal automorphisms of the homogeneous cone \( \hat{S} \subset \mathbb{C}^n \) of a projective submanifold \( S \subset \mathbb{P}^{n-1} \). The main result of [HM05] says that if \( S \) is non-degenerate, then \( \text{aut}(\hat{S})^{(2)} \neq 0 \) if and only if \( S = \mathbb{P}^{n-1} \). [HM05] also proves some partial results on the structure of \( S \) with \( \text{aut}(\hat{S})^{(1)} \neq 0 \).

In a recent joint work with Baohua Fu, we give a complete classification of projective submanifolds \( S \subset \mathbb{P}^{n-1} \) with \( \text{aut}(\hat{S})^{(1)} \neq 0 \). They turn out to be VMRTs of Hermitian symmetric spaces, VMRTs of even and odd symplectic Grassmannians and a finite number of exceptional examples. This result gives a more or less complete description of the infinitesimal symmetries of cone structures and can be regarded as a geometric generalization of E. Cartan’s classification of the prolongations of irreducible Lie algebras.

References


Lisa Jeffrey, Mathematics Department, University of Toronto, Real loci and the based loop group (Joint work with Augustin-Liviu Mare)

Suppose \( (M, \omega) \) is a symplectic manifold with an antisymplectic involution \( \tau \), in other words a smooth map \( \tau: M \to M \) for which \( \tau^2 \) is the identity and \( \tau^*\omega = -\omega \). Suppose \( M \) is equipped with an action of a torus \( T \) satisfying

\[
\tau(tx) = t^{-1}\tau(x)
\]

Lisa Jeffrey, Mathematics Department, University of Toronto, Real loci and the based loop group (Joint work with Augustin-Liviu Mare)
We denote the moment map by $\Phi : M \to \text{Lie}(T)^{\ast}$. (An example is $M = \mathbb{CP}^1$ with $\tau$ given by complex conjugation and the torus being $S^1$ acting by rotation about the vertical axis.) The most frequent source of examples is a space where $\tau$ is given by reflection or complex conjugation.

A theorem of Duistermaat [2] states that

**Theorem 1.** Under the above hypotheses,

1. $\Phi(M) = \Phi(M^\tau)$

2. Using $\mathbb{Z}_2$ coefficients, the $(2j)$-th Betti number of $M$ is equal to the $j$-th Betti number of $M^\tau$.

Duistermaat’s theorem was published one year after the paper of Atiyah [1] which showed that the image of the moment map of a Hamiltonian torus action is a convex polytope, the convex hull of the fixed point set.

Let $G$ be a compact Lie group. In [4], we consider $M = \Omega G$, the based loop group. This is defined as

$$\Omega G = \{ f : S^1 \to G \mid f(\ast) = e \}$$

This is a symplectic manifold [5] equipped with the Hamiltonian group action of $T \times S^1$ where $T$ acts by conjugation ($t : \gamma \mapsto t\gamma g^{-1}$) and $S^1$ acts by rotation

$$(e^{i\theta} \gamma)(e^{i\phi}) = \gamma(e^{i(\theta + \phi)})(e^{i\theta})^{-1}$$

Suppose $G$ is equipped with an involution $\sigma$. Then we define an involution on $\Omega G$ by

$$(\tau(\gamma))(e^{i\phi}) = \sigma(\gamma(e^{i\phi}))$$

We may apply Duistermaat’s theorem to $\Omega G$ in this situation. To obtain the conclusion about the $\mathbb{Z}_2$ cohomology we use a stronger result of Hausmann, Holm and Puppe [3] which applies to conjugation spaces (a particular type of cell complexes, which may have cells in infinitely many dimensions – an example is the infinite-dimensional projective space which forms a classifying space for $S^1$). We prove that $\Omega G$ is a conjugation space, so that we may apply the results of [3] to relate its $\mathbb{Z}_2$ cohomology with that of the fixed point set of the involution. In this situation, Atiyah and Pressley already proved the image of the moment map for the torus action on the based loop group is convex. The fixed point set of the torus action is a discrete set, the integer lattice of the torus (identified with homomorphisms from $U(1)$ to the maximal torus $T$).

Finally we learn that

$$\Omega(G)^\tau = \Omega(G/K)$$

where $K = G^\sigma = \{ k \in G : \sigma(k) = e \}$. So we have shown that

$$H^{2j}(\Omega G; \mathbb{Z}_2) \cong H^j(\Omega(G/K); \mathbb{Z}_2).$$

**References**


Chuu-Lian Terng, UC Irvine, *Isometric immersions of hyperbolic manifolds in Euclidean space, revisited*

A classical theorem of Hilbert in surfaces states that there is no isometric immersion of the complete simply-connected hyperbolic manifold $H^2$ manifold in $R^3$. É. Cartan proved that $H^n$ (i.e., the complete, simply connected, Riemannian manifold with constant sectional curvature $-1$) can be locally embedded in $R^{2n-1}$, but it is still unknown whether there is a global isometric immersion of $H^n$ in $R^{2n-1}$. In this talk, I will explain some of the remarkable properties of the Gauss-Codazzi equations for such immersions. For example, these systems have the following properties:

1. They are involutive exterior differential systems, hence the Cauchy problem can be solved via the Cartan-Kähler theory for local real analytic initial data along a non-characteristic line,
2. They are soliton equations so the Cauchy problem can be solved globally by the Inverse Scattering method for smooth initial data with small norm,
3. They have a loop group symmetry, explicit soliton solutions, bi-Hamiltonian structures, and infinitely many conservation laws.

What we do not know is whether the submanifolds constructed from these global solutions of the Gauss-Codazzi equations have singularities.

Gunther Uhlmann, University of Washington, Seattle, *Boundary rigidity, lens rigidity and travel time tomography*

The boundary rigidity problem consists in determining the Riemannian metric of a compact Riemannian manifold with boundary by measuring the lengths of geodesics joining points of the boundary. The lens rigidity problem consists in determining the Riemannian metric of a compact Riemannian manifold with boundary by measuring the scattering relation or lens relation: We know the point of exit and direction of exit of a geodesic if we know its point of entrance and direction of entrance.

These two problems arise in travel time tomography in which one attempts to determine the index of refraction of a medium by measuring the travel times of waves going through the medium.

We will survey what is known about this problem and some recent results.