

## Texas Geometry and Topology Conference

This is a report on the presentations at the 37th meeting of the Texas Geometry and Topology Conference at the Texas Christian University, March 23-25, 2007. This conference was partially supported by National Science Foundation Grant DMS-0605082 and Texas Christian University. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

### Meeting 37. Texas Christian University, March 23-25, 2007

#### Tara Brendle, Louisiana State University, *Wicket groups and ring groups*

Let  $\mathcal{W}_n$  be the space of all configurations of  $n$  disjoint semicircles in upper halfspace  $\mathbb{R}_+^3$  that meet the plane  $\partial\mathbb{R}_+^3 = \mathbb{R}^2$  orthogonally in their endpoints. We call these semicircles *wickets*, as in the game of croquet. Our main purpose is to study the *wicket group*  $W_n = \pi_1(\mathcal{W}_n)$  and relate it to other better-known groups.

By taking the union of a configuration of wickets in  $\mathcal{W}_n$  with its reflection across the plane  $\mathbb{R}^2$  we obtain a configuration of  $n$  pairwise unlinked circles, so  $\mathcal{W}_n$  embeds in the space  $\mathcal{R}_n$  of all configurations of  $n$  disjoint pairwise unlinked circles, or *rings*, in  $\mathbb{R}^3$ . We will also be interested in the *ring group*  $R_n = \pi_1(\mathcal{R}_n)$ . The inclusion  $\mathcal{W}_n \hookrightarrow \mathcal{R}_n$  induces a homomorphism  $W_n \rightarrow R_n$  which we show is surjective. The group  $R_n$  (as well as finite index subgroups of  $R_n$  obtained by orienting and perhaps also numbering the rings) has been studied before in a number of different guises under such different names as the ‘motion group’ of the unlink [D], [Go], [BL], [R], the ‘symmetric automorphism group’ of a free group [C], [M], [BMMM], the ‘group of loops’ [JMM], the ‘braid-permutation group’ [FRR], and the ‘loop braid group’ [BWC]. Another term that is often used is the ‘string group’.

The connection between  $W_n$  and braid groups arises from associating to each configuration in  $\mathcal{W}_n$  the endpoints of its  $n$  semicircles, a configuration of  $2n$  points in  $\mathbb{R}^2$ . Passing to loops of such configurations, we obtain a homomorphism from  $W_n$  to the braid group  $B_{2n}$ . This homomorphism turns out to be injective, so  $W_n$  embeds as a subgroup of  $B_{2n}$ .

There is a simpler space of wicket configurations, the subspace  $\mathcal{U}_n$  of  $\mathcal{W}_n$  consisting of configurations of wickets lying in planes parallel to a fixed plane, say the  $xz$ -plane. We think of these as ‘untwisted’ wickets, and we set  $U_n = \pi_1(\mathcal{U}_n)$ . We show that the inclusion of  $\mathcal{U}_n$  into  $\mathcal{W}_n$  induces an injection  $U_n \hookrightarrow W_n$  and that  $\mathcal{U}_n$  is a  $K(U_n, 1)$ . We are then able to use a general position argument to produce an explicit finite presentation for  $U_n$ .

A little further effort yields presentations for  $W_n$  and for  $R_n$ , each obtained by adding certain ‘obvious’ generators and relations to that of  $U_n$ . To this end, one technical tool we introduce is the ‘separated wicket’ subspace  $\mathcal{SW}_n$  of  $\mathcal{W}_n$  consisting of configurations of wickets which can be encased in hemispherical ‘shells’ in a particular way. In a similar fashion we define a subspace  $\mathcal{SR}_n$  of  $\mathcal{R}_n$  consisting of separated configurations of rings.

**Theorem.** *The inclusions  $\mathcal{SW}_n \hookrightarrow \mathcal{W}_n$  and  $\mathcal{SR}_n \hookrightarrow \mathcal{R}_n$  are homotopy equivalences.*

There are also ‘floppy’ versions of wicket space and ring space, namely *arc space*  $\mathcal{A}_n$  and *loop space*  $\mathcal{L}_n$ , respectively. More precisely,  $\mathcal{A}_n$  is the space of all configurations of  $n$  disjoint smoothly embedded unknotted arcs in  $\mathbb{R}_+^3$  which intersect  $\partial\mathbb{R}_+^3$  transversely in endpoints;  $\mathcal{L}_n$  is then defined in the obvious way. We show the following.

**Theorem.** *The inclusions  $\mathcal{W}_n \hookrightarrow \mathcal{A}_n$  and  $\mathcal{R}_n \hookrightarrow \mathcal{L}_n$  are homotopy equivalences.*

Generators for  $\mathcal{A}_n = \pi_1(\mathcal{A}_n)$  were found by Hilden [Hi]. Our results give a new derivation of these generators, along with a full set of relations for them. We note that in Hilden’s viewpoint,  $\mathcal{A}_n$  embeds

in a natural way in the so-called *Heegaard subgroup* of the mapping class group of a surface. Also, the group  $\pi_1(\mathcal{L}_n)$  was computed in the 1960s by Dahm [D] by showing that the natural map from  $\pi_1(\mathcal{L}_n)$  to the automorphism group of the free group  $\pi_1(\mathbb{R}^3 - L)$  for  $L \in \mathcal{L}_n$  is injective, with image the so-called symmetric automorphisms, the subgroup generated by automorphisms sending each basis element to a conjugate of itself, along with the automorphisms that permute basis elements or send them to their inverses.

The braid group  $B_n$  is defined as the fundamental group of the space  $\mathcal{C}_n$  of configurations of  $n$  points in  $\mathbb{R}^2$ , and it is an elementary fact that the higher homotopy groups of  $\mathcal{C}_n$  are trivial, so it is a  $K(B_n, 1)$ . Analogously, we show the following.

**Theorem.** *The map  $W_n \rightarrow B_{2n}$  induced by the endpoint map  $\mathcal{W}_n \rightarrow \mathcal{C}_{2n}$  is injective, and  $W_n$  is a  $K(W_n, 1)$ .*

The proof of this theorem is considerably more involved than for  $\mathcal{C}_n$ . Note also that  $\mathcal{R}_n$  is not a  $K(R_n, 1)$ , and indeed there is no finite-dimensional  $K(R_n, 1)$ , since  $R_n$  has torsion, as it contains a copy of the symmetric group  $S_n$ .

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**Richard Canary, University of Michigan, *Introductory bumponomics: the deformation theory of Kleinian groups***

We study the topology of the space  $AH(M)$  of all (marked) hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold  $M$  (with boundary.)

The topology of the interior of  $AH(M)$  is quite simple and has been well-understood since the 1970's. The components are enumerated by (marked) homeomorphism types of compact hyperbolizable 3-manifolds homotopy equivalent to  $M$  and each component is a manifold which is naturally identified with a quotient of a Teichmüller space.

In the last decade it has become clear that the global topology of  $AH(M)$  is quite complicated. Anderson and Canary discovered that components of the interior can “bump” (i.e. have intersecting closures). Anderson, Canary and McCullough completely characterized which components of the interior of  $AH(M)$  bump when  $M$  has incompressible boundary. McMullen gave the first example where a single component of the interior of  $AH(M)$  self-bumps, i.e there is a point in the boundary of the component such that the intersection of any small enough neighborhood of the point with the component is disconnected. Bromberg and Holt showed that self-bumping is a very widespread phenomena, in particular any component of the interior of  $AH(M)$  self-bumps whenever  $M$  contains a primitive essential annulus. Most disturbingly, Bromberg proved that the space of punctured torus groups is not even locally connected.

In this talk, we discuss recent joint work with Brock, Bromberg and Minsky, which shows, in many cases, that the topology is well-behaved at “most” points in the boundary of  $AH(M)$ . We say that a point in the boundary  $\partial AH(M)$  of  $AH(M)$  is *untouchable* if there is no bumping or self-bumping of components at the point. In particular,  $AH(M)$  is locally connected at all untouchable points.

**Theorem 1.** *If  $N \in \partial AH(M)$  has no cusps (i.e. every homotopically non-trivial closed curve in  $N$  is homotopic to a closed geodesic), then  $N$  is untouchable.*

If  $\partial M$  contains no tori, then such points are generic in  $\partial AH(M)$ . Notice that we do not assume in this result that  $\partial M$  has incompressible boundary.

We say that  $N \in \partial AH(M)$  is *quasiconformally rigid* if every component of its conformal boundary is a thrice-punctured sphere.

**Theorem 2.** *If  $N \in \partial AH(M)$  is quasiconformally rigid, then there is no bumping at  $N$ .*

We further say that  $N$  is a *maximal cusp* if  $N$  is quasiconformally rigid and geometrically finite.

**Theorem 3.** *If  $M$  is acylindrical and  $N \in \partial AH(M)$  is a maximal cusp, then there is no self-bumping at  $N$ .*

**Steve Ferry, Rutgers University, *Higson's compactification and the Novikov conjecture***

In 1990, Nigel Higson introduced a very general compactification, reminiscent of the Stone-Cech compactification, for locally compact metric spaces. By 1993, it was known that if the Higson compactification of the universal cover of a closed aspherical manifold is acyclic, then the rational Novikov conjecture follows for the fundamental group of that manifold.

Unfortunately, Dranishnikov and Ferry showed that even for euclidean space, the rational cohomology of the Higson compactification is enormous. In this talk, I will give a survey of the Novikov Conjecture and its implications, and I will report on recent joint work with Dranishnikov and Weinberger that shows the rational Novikov conjecture follows from the mod 2 acyclicity of the Higson compactification. We have also shown that the Higson compactification of  $EG$  is mod  $p$  acyclic for all groups  $G$  such that  $BG$  is a finite complex and  $EG$  is asymptotically finite dimensional. This shows that Higson's compactification can be used to recover an integral Novikov result of Guoliang Yu.

**Greg Friedman, Texas Christian University, *Intersection homology and Poincaré duality on homotopically stratified spaces***

In my talk, I will review some background material on intersection homology and then discuss the extension of Poincaré duality to manifold homotopically stratified spaces using intersection homology.

Intersection homology was introduced by Goresky and MacPherson [9] in order to extend Poincaré duality to *manifold stratified spaces* – spaces that are not manifolds but that are composed of manifolds of various dimensions. This was initially done for *piecewise-linear pseudomanifolds* [9], which include algebraic and analytic varieties<sup>1</sup>, but was soon extended to *topological pseudomanifolds* (Goresky-MacPherson [10]) and *locally conelike topological stratified spaces*, also called *cs-spaces* (Habegger-Saper [11]). The establishment of Poincaré duality for pseudomanifolds has led to the successful study and application of further related invariants. To name just a few: Right in [9], Goresky and MacPherson introduced signatures and *L*-classes for pseudomanifolds with only even codimension strata; Siegel extended signatures and bordism theory to Witt spaces [22]; and various extensions of duality and characteristic classes have been studied by Cappell, Shaneson, and Banagl, in various combinations [7, 1, 2, 4]. For applications of intersection homology in this direction, we refer the reader to [3]; for applications of intersection homology in other fields, we refer the reader to [15].

In [20], Quinn introduced *manifold homotopically stratified spaces* (MHSSs), with the intent to provide “a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds.” In this context of topological group actions on manifolds, MHSSs have been studied by Yan [26], Beshears [5], and Weinberger and Yan [24, 25].<sup>2</sup> But MHSSs also arise in categories with more structure; for example, Cappell and Shaneson showed that they occur as mapping cylinders of maps between smoothly stratified spaces [6]. A surgery theory for MHSSs has been developed by Weinberger [23], and their geometric neighborhood properties have been studied by Hughes, culminating in [13]. In [19], Quinn noted that MHSSs “are defined by local homotopy properties, which seem more appropriate for the study of a homology theory” than the local homeomorphism properties of pseudomanifolds, and he showed that intersection homology is a topological invariant on these spaces, independent of the stratification. A further survey of MHSSs can be found in Hughes and Weinberger [14].

We prove the following Poincaré duality theorem. The stated condition on the MHSS  $X$  is described more fully in the talk but includes MHSSs with compact singular set  $\Sigma$  such that all non-minimal strata of  $X$  have dimension  $\geq 5$ .

**Theorem.** *Let  $X$  be an  $n$ -dimensional MHSS with no codimension one stratum and with sufficiently many local approximate tubular neighborhoods. Let  $\mathcal{O}$  be the orientation sheaf of the  $n$ -manifold  $X - X^{n-2}$ , and let  $\mathcal{E}$  be a local coefficient system of  $\mathbb{F}$ -modules for a field  $\mathbb{F}$ . Let  $\bar{p}$  and  $\bar{q}$  be dual perversities ( $\bar{p}(k) + \bar{q}(k) = k - 2$ ). Then*

$$I^{\bar{q}}H_i^\infty(X; \text{Hom}(\mathcal{E}, \mathbb{F}_{X-X^{n-2}}) \otimes \mathcal{O}) \cong \text{Hom}(I^{\bar{p}}H_{n-i}^c(X; \mathcal{E}; \mathbb{F}).$$

*In particular, if  $X$  is a compact orientable MHSS satisfying the hypotheses of the theorem, we obtain non-singular pairings*

$$I^{\bar{p}}H_i(X; \mathbb{Q}) \otimes I^{\bar{q}}H_{n-i}(X; \mathbb{Q}) \rightarrow \mathbb{Q}.$$

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<sup>1</sup>excluding those with codimension one strata

<sup>2</sup>The application of intersection homology to the study of group actions both on smooth manifolds and on stratified spaces is an active field of research; see, e.g., [12, 21, 16, 8, 18, 17].

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**Robin Hartshorne, University of California at Berkeley, *Gorenstein Liaison of Algebraic Varieties***

In this talk, we consider subschemes  $X$  in a fixed projective space  $\mathbb{P}^n$  over an algebraically closed field  $k$ . Let  $S = k[x_0, x_1, \dots, x_n]$  be the homogeneous coordinate ring of  $\mathbb{P}^n$ , and let  $I_X$  be the homogeneous ideal of forms vanishing on  $X$ . We say that  $X$  is *arithmetically Cohen-Macaulay* (ACM) if the ring  $S/I_X$  is a Cohen-Macaulay ring. We say that  $X$  is *arithmetically Gorenstein* (AG) if  $S/I_X$  is Gorenstein.

We say that schemes  $X_1$  and  $X_2$  of the same dimension are *directly Gorenstein linked* if there is an AG scheme  $Y$  such that  $X_1 \cup X_2 = Y$  as sets, and for  $i \neq j$ , we have  $\mathcal{I}_{X_i, Y} \cong \text{Hom}(\mathcal{O}_{X_j}, \mathcal{O}_Y)$ , where  $\mathcal{I}$  denote the ideal sheaf and  $\mathcal{O}$  denotes the structure sheaf. The equivalence relation of *Gorenstein liaison* is defined by allowing finite chains of direct Gorenstein links.

If the  $X_i$  have codimension 2 in  $\mathbb{P}^n$ , then Gorenstein linkage is the same as the more classical complete intersection linkage. One of the main results in that case is

**Theorem** (Gaeta’s Theorem). *A codimension 2 subspace of  $\mathbb{P}^n$  is ACM if and only if it is the liaison class of a complete intersection (licci for short).*

The proof of Gaeta’s theorem uses the fact that an ACM scheme of codimension 2 is *determinantal*, i.e., defined by the  $r \times r$  minors of an  $r \times (r + 1)$  matrix of homogeneous polynomials. In codimensions greater than or equal to 3, Gaeta’s theorem is no longer true for complete intersection liaison. However, one can raise the same question for Gorenstein liaison:

**Problem.** *Is every ACM subscheme of  $\mathbb{P}^n$  the Gorenstein liaison class of a complete intersection (glicci)?*

This problem remains open at present. However, there is a generalization of Gaeta’s theorem that gives some hope:

**Theorem** (Generalized Gaeta’s Theorem [3], [1]). *Any determinantal scheme (defined by the  $t \times t$  minors of an  $r \times s$  matrix, having the expected codimension) in  $\mathbb{P}^n$  is glicci. Moreover, it can be obtained by ascending Gorenstein biliaisons from a linear variety.*

(I will not give the precise definition of Gorenstein liaison; it is a particular method of constructing Gorenstein liaisons.)

On the other hand, recent examples [2] show that there are ACM codimension 3 schemes that cannot be attained by any ascending Gorenstein biliaison; it is unknown if there examples are glicci.

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### J.M. Landsberg, Texas A & M University, *Differential geometry of submanifolds of projective space*

The theme of this talk is the use of local projective differential geometry to prove theorems in algebraic geometry and representation theory.

A classical topic in the study of surfaces in Euclidean three space  $\mathbb{E}^3$  are the *asymptotic lines* associated to a point on the surface where the Gauss curvature is non-positive. The tangent lines to  $x \in X \subset \mathbb{E}^3$  are the lines that have contact with  $X$  at  $x$  to order at least one, the asymptotic lines  $\widehat{C}_{2,x}$  are the lines having contact to order at least two. One can continue, defining the lines having contact to at least  $k$  at  $x$ ,  $C_{k,x}$  for all  $x$ , and finally  $C_{\infty,x}$  which, in the analytic category are the lines contained in (the completion of)  $X$ . The sets  $C_{k,x}$  are unchanged under arbitrary linear changes of coordinates in  $\mathbb{E}^3$  and the subject is best viewed under the action of this larger group. To prove global results, it is better to move to the compact setting, i.e., submanifolds of projective space, and to have applications to algebraic geometry, one works over the complex numbers.

When we work in the analytic or algebraic category, we would like to recover global information about  $X$  from its infinitesimal geometry at a general point  $x \in X$ . In particular, if one of the  $C_{k,x}$  is different from what is expected, we would like to determine its precise implications for the global geometry of  $X$ .

Let  $X^n \subset \mathbb{C}\mathbb{P}^{n+1}$  be a complex submanifold or algebraic variety. Let  $x \in X$ . From  $C_{1,x} = \mathbb{P}T_x X$  we can determine if  $x$  is a smooth point of  $X$ . If  $x$  is a general point, then  $C_{2,x}$  will be a quadric hypersurface in  $\mathbb{P}T_x X$ . For most hypersurfaces,  $C_{3,x}$  is the intersection of a cubic and quadratic hypersurface in  $\mathbb{P}T_x X$ . In particular, if  $X$  is a surface, we expect  $C_{3,x}$  to be empty at general points. Darboux proved

**Theorem 1** (Darboux). *Let  $X^{2\mathbb{P}^{2+a}}$  be an analytic submanifold and let  $x \in X_{general}$ . If there exists a line  $l$  having contact to order three with  $X$  at  $x$ , then  $l \subset \overline{X}$ . In other words, for surfaces in projective space,  $C_{3,x} = C_x \forall x \in X_{general}$ .*

There are several generalizations of this result in [7, 8]. Here is one of them:

**Theorem 2** ([8]). *Let  $X^n \subset \mathbb{P}^{n+1}$  be an analytic submanifold and let  $x \in X_{general}$ . If  $\Sigma \subseteq C_{k,x}$  is an irreducible component with  $\dim \Sigma > n - k$ , then  $\Sigma \subset C_x$ .*

Here is a rigidity result:

**Theorem 3** ([6, 4, 9, 10]). *Let  $X^n \subset \mathbb{P}W$  be an analytic submanifold and let  $x \in X$  be a general point. Let  $Z = G/P \subset \mathbb{P}V$  be an irreducible compact Hermitian symmetric space in its minimal homogeneous embedding, other than the quadric hypersurface. Assume  $\dim W = \dim V$  and that  $X$  is not contained in a hyperplane. If there exists a linear map  $\ell : T_z Z \rightarrow T_x X$  such that  $C_{2,x,X} = \ell(C_{2,z,Z})$ , then there exists a linear map  $L : V \rightarrow W$  such that  $L(Z) = X$ .*

Long before these results, Fubini [1] proved the quadric hypersurface is rigid to order three.

More general results are known, e.g., for non-minimal embeddings and reducible CHSS, see [4, 6, 9]. The first case of this type of theorem was conjectured by Griffiths and Harris [2] for  $Seg(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$  and solved in [6].

The main new result of this talk is related more closely to representation theory, as it deals with the closed orbit in the adjoint representation of a complex simple Lie algebra. In [12] a new proof of the Killing-Cartan classification of complex simple Lie algebras was given using the local projective differential geometry of these varieties, and in [15, 16] this local geometry was related to the proposed categorical generalizations of Lie algebras of Deligne and Vogel.

**Theorem 4** (Landsberg-Robles). *The adjoint varieties are rigid to order three.*

**Theorem 5** (Landsberg-Robles). *The  $SL_3$ -adjoint variety is not quasi-rigid with respect to the natural exterior differential system induced by the Lie algebra grading.*

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**Paul Loya, Binghamton University, *Gluing formulas for the spectral invariants of Dirac operators***

This is joint work with Jinsung Park. Spectral invariants are, roughly speaking, objects built from the spectrum (or eigenvalues) of Dirac operators defined on Riemannian manifolds. Examples of spectral invariants include the index, eta invariant (also called the Atiyah-Patodi-Singer invariant) [1], and  $\zeta$ -regularized (or Ray-Singer) determinant [8]. Over the past several years there has been interest in understanding the behavior of the spectral invariants of Dirac type operators when the underlying Riemannian manifold is cut into pieces. This has resulted in the search for gluing or pasting formulas for these invariants. In mathematics, such gluing formulas have applications to the Ray-Singer conjecture concerning the equality of torsion invariants, index theory on manifolds with corners and gluing formulas for Dirac determinant line bundles. In mathematical physics, gluing formulas play roles in Donaldson, Floer, and Seiberg-Witten theory and especially in the development of topological quantum field theory where pasting laws for  $\zeta$ -determinants are desired.

The basic statement of the “gluing problem” is as follows: Given a partitioned compact manifold  $M = M_- \cup M_+$  into manifolds with boundaries, describe the spectral invariants of Dirac operators on the manifold  $M$  in terms of the invariants on the manifolds with boundary  $M_{\pm}$  with suitable boundary conditions. The gluing problem for the index was the first to be solved. This was settled by Atiyah, Patodi, and Singer [1], and the solution used the local nature of the index. The *Bojarski Conjecture*, later proved by Booß-Bavnbek and Wojciechowski [3], gives a general gluing formula for the index in terms of the Fredholm index of the pair of Cauchy data spaces from  $M_{\pm}$ . Next, the gluing problem for the eta invariant was solved by Bunke [4] and others. The main difficulty in this case has to do with the nonlocal nature of the eta invariant, in contrast to the local nature of the index. However, the variation of the eta invariant is local, and because of this locality a variety of formulas and proofs for the gluing problem have been found (many modulo  $\mathbb{Z}$ ); see the survey articles by Bleecker and Booß-Bavnbek [2] and Mazzeo and Piazza [7] for more references on this topic. Of these solutions, the one by Kirk and Lesch [5] is the most complete and general and, as with Booß-Bavnbek and Wojciechowski’s [3] solution to the index problem, involves the two Cauchy data spaces from  $M_{\pm}$ .

In [6], Jinsung Park and I derive the gluing formula for the  $\zeta$ -determinant. As with Booß-Bavnbek and Wojciechowski’s [3] solution to the index problem and Kirk and Lesch’s [5] solution to the eta problem, the solution involves the two Cauchy data spaces from  $M_{\pm}$ . Moreover, the gluing formula for the eta invariant and the  $\zeta$ -determinant are proved simultaneously because it turns out that the gluing formulas for the eta invariant and the  $\zeta$ -determinant are not entirely separate formulas, but are really just two aspects of one formula — *the phase and modulus* — of the same unified formula, which involves the two Cauchy data spaces from  $M_{\pm}$ .

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**Christina Sormani, CUNY Graduate Center and Lehman College, *The covering spectrum of a Riemannian manifold***

With the outstanding advances in mathematics over the past few decades it has become increasingly difficult for graduate students to begin a research program. Yet facing the current job market without a published paper can be a difficult task for a young PhD. In my talk, I will present a new geometrically intuitive concept which can be understood by any student with a strong background in metric spaces and topology: two fields often taught at the undergraduate level in the United States. One of the papers I am reporting on, “Convergence and the Length Spectrum” (to appear AIM 2007), can be easily read by second year graduate students and includes a list of 30 problems. Most of these problems are meant to be accessible to a graduate student and yet advanced enough to publish. One expects the solutions are likely to be effectively described with the graphics accompanying the necessary equations; so they lend themselves immediately to job talks and interviews and will give the graduate students the opportunity to display their teaching ability on such occasions. This is especially important for graduate students planning careers at teaching colleges and community colleges across the country.