Name________________________________

MATH 172 Honors   Exam 1   Spring 2024

Section 200   Solutions   P. Yasskin

Points indicated. Part credit possible. Show all work.

1	/8	5	/8	9	/12
2	/8	6	/8	10	/12
3	/8	7	/8	11	/8
4	/8	8	/8	12	/8
Total			/104		

1. (8 points) Estimate the integral $\int_0^8 x^2\,dx$ Approximate integral using a Riemann sum with 4 equal intervals and left endpoints.
Is this an over estimate or under estimate. Why? Your answer should be based on the concepts of increasing, decreasing, concave up or concave down.

Solution: $f(x) = x^2$ $\quad \Delta x = \dfrac{8-0}{4} = 2$ $\quad$ The left endpoints are $x_i = 0,2,4,6.$

The function values are $f(x_i) = 0,4,16,36.$ So the Riemann sum is

$$\sum_{i=1}^{4} f(x_i)\Delta x = (0+4+16+36)2 = 112$$

Since the function x^2 is increasing, the value at the left endpoint is a minimum on each interval. So the sum is an under estimate.

2. (8 points) Consider the area below the graph of $y = x^3$ above the x-axis. Find the number c so that the area between $x = 0$ and $x = c$ is equal to the area between $x = c$ and $x = 4$.

Solution: The first area is $A_1 = \int_0^c x^3\,dx = \left[\dfrac{x^4}{4}\right]_0^c = \dfrac{c^4}{4}.$

The second area is $A_2 = \int_c^4 x^3\,dx = \left[\dfrac{x^4}{4}\right]_c^4 = 4^3 - \dfrac{c^4}{4}.$

We solve $\dfrac{c^4}{4} = 4^3 - \dfrac{c^4}{4}$: $\quad 2\dfrac{c^4}{4} = 4^3 \quad c^4 = \dfrac{4^4}{2} \quad c = \dfrac{4}{\sqrt[4]{2}}$

3. (8 points) Compute $\int_2^4 \dfrac{x+1}{(x^2+2x)^2}\,dx.$ Simplify to a rational number.

Solution: Substitute $u = x^2 + 2x$. Then $du = (2x+2)\,dx$ and $\dfrac{1}{2}du = (x+1)dx$. So

$$\int_2^4 \frac{x+1}{(x^2+2x)^2}\,dx = \frac{1}{2}\int_8^{24}\frac{1}{u^2}\,du = \frac{1}{2}\left[\frac{-1}{u}\right]_8^{24} = \frac{1}{2}\left(\frac{-1}{24}+\frac{1}{8}\right) = \frac{1}{2}\left(\frac{3-1}{24}\right) = \frac{1}{24}$$

4. (8 points) Compute $\int_0^{\pi/4}(\sec^4\theta - \tan^2\theta\sec^2\theta)\,d\theta.$ Evaluate all trig functions.

Solution: Since $\sec^2\theta - \tan^2\theta = 1$,

$$\int_0^{\pi/4}(\sec^4\theta - \tan^2\theta\sec^2\theta)\,d\theta = \int_0^{\pi/4}(\sec^2\theta - \tan^2\theta)\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^2\theta\,d\theta = \Big[\tan\theta\Big]_0^{\pi/4}$$
$$= \tan\frac{\pi}{4} - \tan 0 = 1$$

5. (8 points) Compute $\int 2x \arctan x \, dx$.

Solution: Integrate by parts with $\begin{array}{ll} u = \arctan x & dv = 2x\,dx \\ du = \dfrac{1}{1+x^2}dx & v = x^2 \end{array}$

$$I = \int 2x \arctan x \, dx = x^2 \arctan x - \int \frac{x^2}{1+x^2} dx = x^2 \arctan x - \int \frac{x^2+1-1}{1+x^2} dx$$
$$= x^2 \arctan x - \int 1 - \frac{1}{1+x^2} dx = x^2 \arctan x - x + \arctan x + C$$

6. (8 points) Compute $\int \sin(2\theta)\cos^2(\theta)\,d\theta$

Solution:
$$I = \int \sin(2\theta)\cos^2(\theta)\,d\theta = \int 2\sin\theta\cos\theta\cos^2\theta d\theta = \int 2\cos^3\theta\sin\theta d\theta = -2\frac{\cos^4\theta}{4} + C = -\frac{\cos^4\theta}{2} + C$$

Or: $I = \int \sin(2\theta)\cos^2(\theta)\,d\theta = \int \dfrac{1+\cos(2\theta)}{2}\sin(2\theta)\,d\theta$

Let $u = \cos(2\theta)$. Then $du = -2\sin(2\theta)\,d\theta$ and $-\frac{1}{2}du = \sin(2\theta)\,d\theta$. So

$$I = -\frac{1}{2}\int \frac{1+u}{2}du = -\frac{1}{4}\left(u + \frac{u^2}{2}\right) + C = -\frac{1}{4}\cos(2\theta) - \frac{1}{8}\cos^2(2\theta) + C$$

7. (8 points) Compute $\int e^{2x}\sin 4x\,dx$.

Solution: Integrate by parts with $\begin{array}{ll} u = \sin 4x & dv = e^{2x}dx \\ du = 4\cos 4x\,dx & v = \frac{1}{2}e^{2x} \end{array}$

$I = \frac{1}{2}e^{2x}\sin 4x - 2\int e^{2x}\cos 4x\,dx$ Parts again $\begin{array}{ll} u = \cos 4x & dv = e^{2x}dx \\ du = -4\sin 4x\,dx & v = \frac{1}{2}e^{2x} \end{array}$

$$I = \frac{1}{2}e^{2x}\sin 4x - 2\left[\frac{1}{2}e^{2x}\cos 4x + 2\int e^{2x}\sin 4x\,dx\right] = \frac{1}{2}e^{2x}\sin 4x - e^{2x}\cos 4x - 4I$$

$5I = \frac{1}{2}e^{2x}\sin 4x - e^{2x}\cos 4x \qquad I = \frac{1}{10}e^{2x}\sin 4x - \frac{1}{5}e^{2x}\cos 4x + C$

8. (8 points) Compute $\int \dfrac{\sqrt{x^2-4}}{x}dx$.

Solution: Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-4}}{x}dx = \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta}2\sec\theta\tan\theta\,d\theta = 2\int\sqrt{\sec^2\theta - 1}\,\tan\theta\,d\theta = 2\int\tan^2\theta\,d\theta$$
$$= 2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + C$$

Since $\sec\theta = \frac{x}{2}$ we have $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{x^2}{4} - 1} = \dfrac{\sqrt{x^2-4}}{2}$. So

$$\int \frac{\sqrt{x^2-4}}{x}dx = \sqrt{x^2-4} - 2\,\mathrm{arcsec}\,\frac{x}{2} + C$$

9. (12 points) A bar of length $\frac{\pi}{4}$ m has linear density $\delta = \sin x$ kg/m where x is measured from one end.

a. Find the total mass of the bar.

Solution: $M = \int \delta\, dx = \int_0^{\pi/4} \sin x\, dx = \left[-\cos x\right]_0^{\pi/4} = -\frac{1}{\sqrt{2}} - -1 = \frac{\sqrt{2}-1}{\sqrt{2}}$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\, dx = \int_0^{\pi/2} x \sin x\, dx$ Integrate by parts with $u = x \quad dv = \sin x\, dx$, $du = dx \quad v = -\cos x$

$$M_1 = \int_0^{\pi/4} x \sin x\, dx = -x\cos x + \int \cos x\, dx = \left[-x\cos x + \sin x\right]_0^{\pi/4} = -\frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$$

$$\bar{x} = \frac{M_1}{M} = \frac{4-\pi}{4\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}-1} = \frac{4-\pi}{4\left(\sqrt{2}-1\right)}$$

10. (12 points) A race car starts from rest ($x = 0$ and $v = 0$ at $t = 0$) and has acceleration $a = \frac{t}{1+t}$.

a. Find its velocity at time t.

Solution: $v(t) = \int a\, dt = \int \frac{t}{1+t}\, dt$ Let $u = 1 + t \quad du = dt \quad t = u - 1$

$$v(t) = \int \frac{u-1}{u}\, du = u - \ln|u| + C = 1 + t - \ln(1+t) + C \qquad v(0) = 1 + C = 0 \qquad C = -1$$

$$v(t) = t - \ln(1+t)$$

b. Find its position at time t.

Solution: $x(t) = \int v\, dt = \int t - \ln(1+t)\, dt = \frac{t^2}{2} - [(1+t)\ln(1+t) - (1+t)] + K$

$$x(0) = \frac{0^2}{2} - [(1+0)\ln(1+0) - (1+0)] + K = 1 + K = 0 \qquad K = -1$$

$$x(t) = \frac{t^2}{2} - (1+t)\ln(1+t) + t$$

11. (8 points) Find the arclength of the curve $y = \frac{e^x + e^{-x}}{2}$ for $0 \le x \le 1$.

HINT: Look for a perfect square.

Solution: $\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{1}{4}(e^x - e^{-x})^2}\, dx = \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}\, dx = \sqrt{\frac{1}{4}(e^{2x} + 2 + e^{-2x})}\, dx$$

$$= \sqrt{\frac{1}{4}(e^x + e^{-x})^2}\, dx = \frac{e^x + e^{-x}}{2}\, dx$$

$$L = \int ds = \int_0^1 \frac{e^x + e^{-x}}{2} dx = \left[\frac{e^x - e^{-x}}{2}\right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

12. (8 points) The curve $\vec{r}(t) = (t^2, 6t)$ between $t = 0$ and $t = 4$ is rotated about the x-axis, find the surface area swept out.

Solution: $r = y = 6t \qquad \frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 6$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{4t^2 + 36}\, dt = 2\sqrt{t^2 + 9}\, dt$$

$$A = \int 2\pi r\, ds = \int_0^4 2\pi 6t\sqrt{4t^2 + 36}\, dt = 12\pi \int_0^4 2t\sqrt{t^2 + 9}\, dt \qquad u = t^2 + 9 \qquad du = 2t\, dt$$

$$A = 12\pi \int_9^{25} u^{1/2}\, du = 12\pi\left[\frac{2u^{3/2}}{3}\right]_9^{25} = 8\pi(125 - 27) = 8 \cdot 98\pi = 784\pi$$