Name________________________________

MATH 172 Honors Exam 2 Spring 2024

Section 200 Solutions P. Yasskin

Points indicated. Part credit possible. Show all work.

1	/?	5	/10	9	/ 4
2	/8	6	/10	10	/ 8
3	/8	7	/10	11	/10
4	/8	8	/10	12	/10
Total			/96+?		

1. (? points) Circle each term in the general partial fraction expansion for $p(x) = \dfrac{5x}{(x-2)(x^4-16)}$.

(2 points for each correct term. −2 points for each incorrect term.)

$$\frac{A}{x+2} \qquad \frac{C}{x-2} \qquad \frac{Ex+F}{x^2-4} \qquad \frac{Ix+J}{x^2+4}$$
$$\frac{B}{(x+2)^2} \qquad \frac{D}{(x-2)^2} \qquad \frac{Gx+H}{(x^2-4)^2} \qquad \frac{Kx+L}{(x^2+4)^2}$$

Solution: We first need to factor the denominator and group factors:

$$p(x) = \frac{5x}{(x-2)(x^2+4)(x^2-4)} = \frac{5x}{(x-2)(x^2+4)(x+2)(x-2)} = \frac{5x}{(x+2)(x-2)^2(x^2+4)}$$

$$p(x) = \frac{A}{x+2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{Ix+J}{x^2+4}$$

2. (8 points) Find the coefficients in the partial fraction expansion

$$\frac{48x^2}{(x^4-81)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{Cx+D}{x^2+9}$$

Write the final expansion.

Solution: Clear the denominator: $48x^2 = A(x+3)(x^2+9) + B(x-3)(x^2+9) + (Cx+D)(x^2-9)$

$x = 3$: $\quad 48 \bullet 9 = A(6)(18) \qquad A = 4$

$x = -3$: $\quad 48 \bullet 9 = B(-6)(18) \qquad B = -4$

$x = 1$:

$$48 = A(4)(10) + B(-2)(10) + (C+D)(-8) = 4(4)(10) - 4(-2)(10) - 8(C+D) = 240 - 8(C+D)$$

$x = -1$:

$$48 = A(2)(10) + B(-4)(10) + (-C+D)(-8) = 4(2)(10) - 4(-4)(10) - 8(-C+D) = 240 - 8(-C+D)$$

$$6 = 30 - C - D \qquad 6 = 30 + C - D \qquad C + D = 24 \qquad C - D = -24$$

Add: $C = 0$ Subtract: $2D = 48 \qquad D = 24$

$$\frac{48x^2}{(x^4-81)} = \frac{4}{x-3} + \frac{-4}{x+3} + \frac{24}{x^2+9}$$

3. (8 points) Given that $\dfrac{3x^3+18x^2-81x+162}{x(x^2+9)(x-3)^2} = \dfrac{2}{x} + \dfrac{3}{(x-3)^2} + \dfrac{-2x}{x^2+9}$

compute $I = \displaystyle\int \frac{3x^3+18x^2-81x+162}{x(x^2+9)(x-3)^2}\,dx$.

Solution: $I = \displaystyle\int \frac{2}{x} + \frac{3}{(x-3)^2} + \frac{-2x}{x^2+9}\,dx = 2\ln|x| - \frac{3}{x-3} - \ln|x^2+9| + C$

4. (8 points) Compute the improper integral $I = \displaystyle\int_0^1 \frac{2x}{\sqrt{1-x^2}}\,dx$.

Solution: Let $u = 1-x^2$. Then $du = -2x\,dx$. We have $x=0$ at $u=1$ and $x=1$ at $u=0$.

$$I = \int_1^0 \frac{-1}{\sqrt{u}}\,du = \Big[-2\sqrt{u}\,\Big]_1^0 = 0 - \left(-2\sqrt{1}\right) = 2$$

5. (10 points) The area between $x = 1-(y-1)^2$ and the y-axis is rotated about the x-axis. Find the volume swept out.

Solution: We do a y integral. Slices are horizontal and rotate into cylinders.
The radius is $r = y$ and the height is $h = x = 1-(y-1)^2 = 2y-y^2$.
The curve crosses the y-axis at $y=0$ and $y=2$. So the volume is

$$V = \int_0^2 2\pi rh\,dy = \int_0^2 2\pi y(2y-y^2)\,dy = \int_0^2 2\pi(2y^2-y^3)\,dy = 2\pi\left[2\frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi\left(2\frac{2^3}{3} - \frac{2^4}{4}\right) = 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{8}{3}\pi$$

6. (10 points) The area between $x = 1-(y-1)^2$ and the y-axis is rotated about the y-axis. Find the volume swept out.

Solution: We do a y integral. Slices are horizontal and rotate into disks.
The radius is $r = x = 1-(y-1)^2 = 2y-y^2$.
The curve crosses the y-axis at $y=0$ and $y=2$. So the volume is

$$V = \int_0^2 \pi r^2\,dy = \int_0^2 \pi(2y-y^2)^2\,dy = \pi\int_0^2 (4y^2-4y^3+y^4)\,dy = \pi\left[4\frac{y^3}{3} - y^4 + \frac{y^5}{5}\right]_0^2$$

$$= \pi\left(4\frac{2^3}{3} - 2^4 + \frac{2^5}{5}\right) = \pi 2^4\left(\frac{2}{3} - 1 + \frac{2}{5}\right) = \pi 2^4\,\frac{10-15+6}{15} = \frac{16}{15}\pi$$

7. (10 points) Solve the initial value problem $(1+x^2)\dfrac{dy}{dx} + 2xy = 4x^3$ with $y(1) = 2$.

Solution: The equation is linear. The standard form is $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{4x^3}{1+x^2}$.

We identify $P = \dfrac{2x}{1+x^2}$. The integration factor is $I = e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$.

We multiply the standard form by the integrating factor to get $(1+x^2)\dfrac{dy}{dx} + 2xy = 4x^3$.

(Back where we started.)

We identify the left as a product, $\dfrac{d}{dx}((1+x^2)y) = 4x^3$ and integrate,

$(1+x^2)y = \int 4x^3\,dx = x^4 + C$.

To find C we plug in $x = 1$ and $y = 2$ to get $4 = 1 + C$ or $C = 3$.

We substitute back: $(1+x^2)y = x^4 + 3$ and solve for $y = \dfrac{x^4+3}{1+x^2}$.

8. (10 points) Solve the initial value problem $3\dfrac{dy}{dx} + \dfrac{2x}{y^2} = \dfrac{1}{y^2}$ with $y(2) = 1$.

Give the explicit solution.

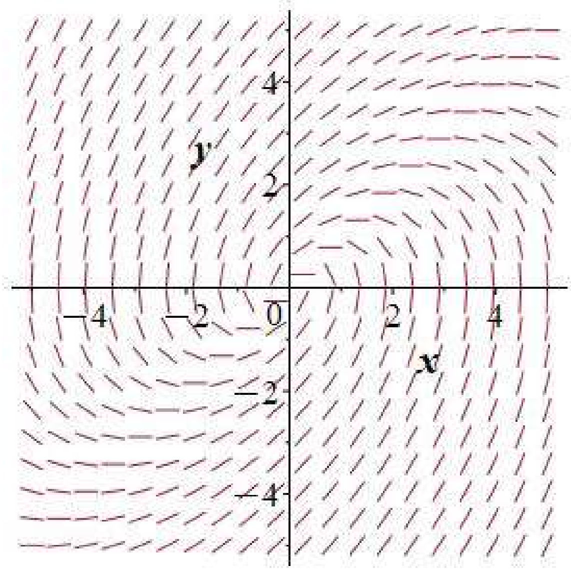
Solution: We write the equation in separable form. $\dfrac{dy}{dx} = \dfrac{1-2x}{3y^2}$.

We separate and integrate. $\int 3y^2\,dy = \int (1-2x)\,dx$. So $y^3 = x - x^2 + C$.

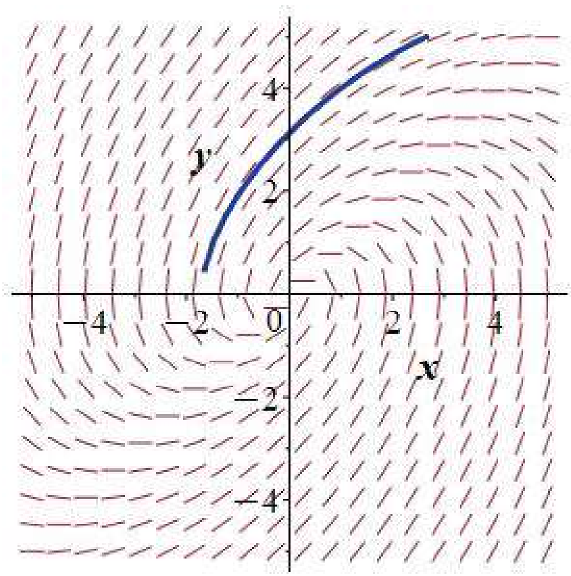
To find C we plug in $x = 2$ and $y = 1$ to get $1 = 2 - 4 + C$ or $C = 3$.

We substitute back: $y^3 = x - x^2 + 3$ and solve for $y = \sqrt[3]{x - x^2 + 3}$.

9. (4 points) At the right is the slope field of the differential equation $\frac{dy}{dx} = \frac{y-x}{y}$.
On this plot sketch the solution curve satisfying the initial condition $y(1) = 4$.



Solution: The curve passes through $(1,4)$ and is always tangent to the line segments.



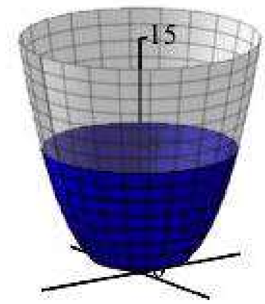
10. (8 points) The area between the curve $y = \frac{1}{x^p}$ and the x-axis for $1 \le x \le \infty$ is rotated about the x-axis, sweeping out a volume.
For which values of p is the volume finite? Be sure to check the border line case.

Solution: We do an x integral using disks. The radius is $r = y = \frac{1}{x^p}$.
So the area is $A = \pi r^2 = \pi \frac{1}{x^{2p}}$.
So the volume is

$$V = \int_1^\infty \pi \frac{1}{x^{2p}}\, dx = \pi \int_0^\infty x^{-2p}\, dx = \pi \left[\frac{x^{-2p+1}}{-2p+1} \right]_1^\infty = \lim_{x\to\infty} \frac{\pi x^{-2p+1}}{-2p+1} - \frac{\pi}{-2p+1}$$

If $-2p+1 < 0$, i.e. $p > \frac{1}{2}$ then the volume is finite, $V = \frac{\pi}{2p-1}$.
If $-2p+1 > 0$, i.e. $p < \frac{1}{2}$ then the volume is infinite.
In the borderline case, if $p = \frac{1}{2}$, then the volume is $V = \int_1^\infty \pi \frac{1}{x}\, dx = \left[\pi \ln x \right]_0^\infty = \infty$

11. (10 points) A water tank is formed by rotating the curve $y = x^3$ for $y \le 15$ meters about the y-axis. It is filled to a depth of 8 meters. Find the volume of water in the tank. HINT: Use horizontal slices.



Solution: The slice at height y is a circle of radius $r = x = y^{1/3}$.
So the cross sectional area is $A = \pi r^2 = \pi y^{2/3}$. And the volume is

$$V = \int_0^8 A\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\frac{3y^{5/3}}{5}\right]_0^8 = \frac{3\pi}{5}8^{5/3} = \frac{3\pi}{5}2^5 = \frac{96}{5}\pi$$

12. (10 points) For the water tank described in the previous problem, find the work done to pump the water out the top of the tank.
Give your answer as a multiple of δg where δ is the density of water and g is the acceleration of gravity.

Solution: A horizontal slice at height y has area $A = \pi r^2 = \pi y^{2/3}$.
Its volume is $dV = A\,dy = \pi y^{2/3}\,dy$ So its weight is $dF = \delta g\,dV = \delta g \pi y^{2/3}\,dy$.
The water is lifted a distance $D = 15 - y$. So the work done is

$$W = \int D\,dF = \int_0^8 (15 - y)\delta g \pi y^{2/3}\,dy = \delta g \pi \int_0^8 (15y^{2/3} - y^{5/3})\,dy$$

$$= \delta g \pi\left[15\frac{3y^{5/3}}{5} - \frac{3y^{8/3}}{8}\right]_0^8 = \delta g \pi\left(9 \cdot 8^{5/3} - 3\frac{8^{8/3}}{8}\right) = \delta g \pi(9 \cdot 2^5 - 3 \cdot 2^5) = 192\delta g \pi$$