1-8	/48	11	/12
9	/20	12	/12
10	/12	Total	/104

Name________________________________

MATH 221 Exam 1, Version A Spring 2024

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Multiple Choice: (6 points each. No part credit.)

1. A point has spherical coordinates $(\rho,\phi,\theta) = \left(6, \frac{\pi}{6}, \frac{\pi}{4}\right)$. Find its cylindrical coordinates.

a. $(r,\theta,z) = \left(3, \frac{\pi}{6}, 3\sqrt{3}\right)$

b. $(r,\theta,z) = \left(3, \frac{\pi}{4}, 3\sqrt{3}\right)$ Correct

c. $(r,\theta,z) = \left(3, \frac{\pi}{6}, 6\sqrt{3}\right)$

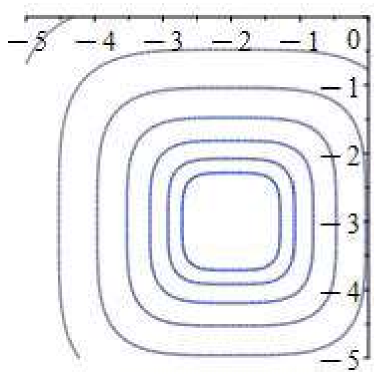
d. $(r,\theta,z) = \left(6\sqrt{3}, \frac{\pi}{6}, 3\right)$

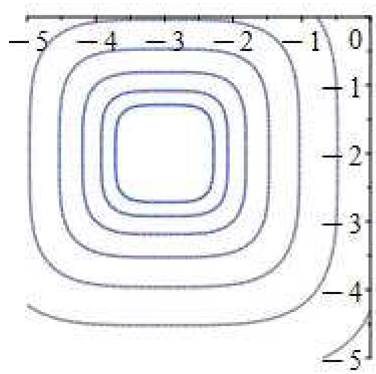
e. $(r,\theta,z) = \left(3\sqrt{3}, \frac{\pi}{4}, 3\right)$

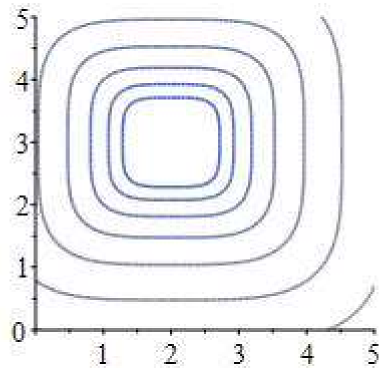
Solution:

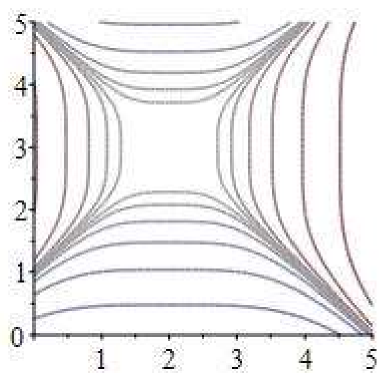
$z = \rho\cos\phi = 6\cos\frac{\pi}{6} = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$ $\quad r = \rho\sin\phi = 6\frac{1}{2} = 3$ $\quad (r,\theta,z) = \left(3, \frac{\pi}{4}, 3\sqrt{3}\right)$

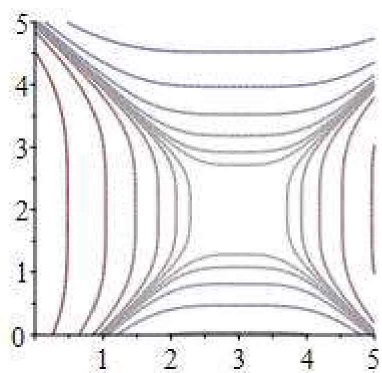
2. Which of the following is the contour plot of the function $f(x,y) = (x-2)^4 + (y-3)^4$?

a. 

b. 

c. 

d. 

e. 

Solution: The function is everywhere positive except at $(x,y) = (2,3)$ where it is 0.
So $(2,3)$ is a local minimum. Contours circle around a minimum or maximum. So (**c**) is the answer.

3. A hiker starts at the point $P = (4,2)$, travels along the vector $\vec{a} = \langle 2,-2\rangle$, then along the vector $\vec{b} = \langle 1,3\rangle$ and finally along the vector $\vec{c} = \langle -1,2\rangle$. Along what vector should the hiker travel to get back to the starting point P?

a. $\langle -2,-3\rangle$ Correct
b. $\langle -6,-5\rangle$
c. $\langle 6,5\rangle$
d. $\langle 2,3\rangle$
e. $\langle 2,-1\rangle$

Solution: In total the hiker travels along the vector $\vec{v} = \vec{a} + \vec{b} + \vec{c} = \langle 2,-2\rangle + \langle 1,3\rangle + \langle -1,2\rangle = \langle 2,3\rangle$. To get back the hiker must travel along $-\vec{v} = \langle -2,-3\rangle$. The point P is irrelevant.

4. For what value of p is $\vec{u} = \langle p,5,3\rangle$ perpendicular to $\vec{v} = \langle 2,1,p\rangle$?

a. $p = -2$
b. $p = -1$ Correct
c. $p = 0$
d. $p = 1$
e. $p = 2$

Solution: They are perpendicular if their dot product is 0.
$\vec{u} \cdot \vec{v} == \langle p,5,3\rangle \cdot \langle 2,1,p\rangle = 2p + 5 + 3p = 5p + 5 = 0$ for $p = -1$

5. Find the volume of the parallelepiped with edge vectors $\vec{a} = \langle 4,2,0\rangle$, $\vec{b} = \langle 1,0,-3\rangle$ and $\vec{c} = \langle 0,-1,2\rangle$.

a. -16
b. -12
c. 8
d. 12
e. 16 Correct

Solution: $\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 4 & 2 & 0 \\ 1 & 0 & -3 \\ 0 & -1 & 2 \end{vmatrix} = 4(0-3) - 2(2-0) + 0(-1-0) = -16$

Volume $= \left|\vec{a} \cdot \vec{b} \times \vec{c}\right| = |-16| = 16$

6. If $\hat{T}$ points Up and $\hat{B}$ points NorthEast, in what direction does $\hat{N}$ point?

a. SouthEast Correct
b. SouthWest
c. NorthWest
d. Down

Solution: $\hat{N} = \hat{B} \times \hat{T}$ If the fingers of your right hand point NorthEast along $\hat{B}$ and the palm faces Up toward $\hat{T}$, then your thumb points SouthEast along $\hat{N}$.

7. Which of the following is a plane perpendicular to the line $(x,y,z) = (1+3t, 3+2t, 4-t)$?

a. $3x - 2y - z = 3$
b. $-3x + 2y + z = 2$
c. $x + 3y + 4z = 5$
d. $3x + 2y - z = 7$ Correct
e. $x - 3y + 4z = 5$

Solution: The normal to the plane is the direction of the line: $\vec{N} = \vec{v} = \langle 3, 2, -1 \rangle$. Any plane with this normal has the form $\vec{N} \cdot X = \vec{N} \cdot P$, or $3x + 2y - z = D$.

8. Classify the quadratic surface: $-x^2 + 2x + y^2 + 4y - 2z^2 + 12z = 14$

a. Hyperbolic Paraboloid opening up in the x-direction and down in the y-direction
b. Hyperbolic Paraboloid opening up in the y-direction and down in the x-direction
c. Hyperboloid of 1 sheet Correct
d. Hyperboloid of 2 sheets
e. Cone

Solution: We complete the squares on x, y and z:
$-(x^2 - 2x) + (y^2 + 4y) - 2(z^2 - 6z) = 14$
$-(x^2 - 2x + 1) + (y^2 + 4y + 4) - 2(z^2 - 6z + 9) = 14 - 1 + 4 - 18 = -1$
$-(x-1)^2 + (y+2)^2 - 2(z-3)^2 = -1$
To get the standard form (with a 1 on the right), we multiply by -1:
$(x-1)^2 - (y+2)^2 + 2(z-3)^2 = 1$
This is a hyperboloid of 1 sheet.

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (20 pts) Consider the twisted cubic $\vec{r} = (t^3, 3t^2, 6t)$. Compute each of the following.
Note: $t^4 + 4t^2 + 4 = (t^2+2)^2$

a. (6 pts) Arc length between $(0,0,0)$ and $(1,3,6)$.

Solution: $\vec{v} = \langle 3t^2, 6t, 6\rangle \qquad |\vec{v}| = \sqrt{9t^4 + 36t^2 + 36} = 3\sqrt{t^4 + 4t^2 + 4} = 3\sqrt{(t^2+2)^2} = 3(t^2+2)$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 3(t^2+2)\,dt = 3\left[\frac{t^3}{3} + 2t\right]_0^1 = 3\left[\frac{1}{3} + 2\right] = 7$$

b. (6 pts) Curvature $\kappa = \dfrac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$.
HINT: Factor out an 18^2.

Solution: $\vec{a} = \langle 6t, 6, 0\rangle \qquad \vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 & 6t & 6 \\ 6t & 6 & 0 \end{vmatrix} = \langle -36, 36t, 18t^2 - 36t^2\rangle = \langle -36, 36t, -18t^2\rangle$

$$|\vec{v}\times\vec{a}| = \sqrt{36^2 + 36^2t^2 + 18^2t^4} = 18\sqrt{4 + 4t^2 + t^4} = 18\sqrt{(t^2+2)^2} = 18(t^2+2)$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{18(t^2+2)}{3^3(t^2+2)^3} = \frac{2}{3(t^2+2)^2}$$

c. (4 pts) Tangential acceleration, a_T.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_T = \dfrac{d}{dt}|\vec{v}| = \dfrac{d}{dt}3(t^2+2) = 6t$

d. (4 pts) Normal acceleration, a_N.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_n = \kappa|\vec{v}|^2 = \dfrac{2}{3(t^2+2)^2}3^2(t^2+2)^2 = 6$

10. (12 pts) Write the vector, $\vec{a} = \langle 5, -3, 1\rangle$, as a sum of two vectors $\vec{p}$ and $\vec{q}$, where $\vec{p}$ is parallel to $\vec{b} = \langle 6,2,4\rangle$ and $\vec{q}$ is perpendicular to $\vec{b}$.

Solution: $\vec{p} = \text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\vec{b} = \dfrac{30-6+4}{36+4+16}\langle 6,2,4\rangle = \dfrac{28}{56}\langle 6,2,4\rangle = \dfrac{1}{2}\langle 6,2,4\rangle = \langle 3,1,2\rangle$

$\vec{q} = \vec{a} - \vec{p} = \langle 5,-3,1\rangle - \langle 3,1,2\rangle = \langle 2,-4,-1\rangle$

We check $\vec{p} + \vec{q} = \langle 3,1,2\rangle + \langle 2,-4,-1\rangle = \langle 5,-3,1\rangle = \vec{a}$

$\vec{p} \parallel \vec{b}$ because $\vec{p} = \langle 3,1,2\rangle$ is a multiple of $\vec{b} = \langle 6,2,4\rangle$.

$\vec{q} \perp \vec{b}$ becaues $\vec{q}\cdot\vec{b} = 12 - 8 - 4 = 0$

11. (12 pts) Consider the helix $\vec{r}(\theta) = \langle 4\cos\theta, 4\sin\theta, 3\theta \rangle$ for $0 \le \theta \le 2\pi$.

a. Find its mass, if its linear density is $\delta(x,y,z) = z$.

Solution: The velocity is $\vec{v} = \langle -4\sin\theta, 4\cos\theta, 3 \rangle$.
The speed is $|\vec{v}| = \sqrt{16\sin^2\theta + 16\cos^2\theta + 9} = 5$.
The density along the curve is $\delta = z = 3\theta$. So the mass is

$$M = \int_0^{2\pi} \delta |\vec{v}|\, dt = \int_0^{2\pi} 3\theta 5\, d\theta = 15\left[\frac{\theta^2}{2}\right]_0^{2\pi} = 15(2\pi^2) = 30\pi^2$$

b. Find the work done to push a bead along the helix if the force is $\vec{F} = \langle -2y, 2x, 0 \rangle$.

Solution: Along the curve the force is $\vec{F} = \langle -8\sin\theta, 8\cos\theta, 0 \rangle$. So the work is

$$W = \int_0^{2\pi} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (32\sin^2\theta + 32\cos^2\theta)\, d\theta = \int_0^{2\pi} 32\, d\theta = 64\pi$$

12. (12 pts) Consider the planes:

$$P_1 : \quad x + y - z = 3$$
$$P_2 : \quad x + 3y + 3z = 5$$

Determine if they are parallel or intersecting. If they intersect, find the line of intersection.
You MUST explain why they are or are not parallel.

Solution: The normal vectors are $\vec{N}_1 = \langle 1, 1, -1 \rangle$ and $\vec{N}_2 = \langle 1, 3, 3 \rangle$.
Since these are not multiples of each other, the planes are not parallel.
The direction of the line of intersection is the cross product of the normals:

$$\vec{v} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 3 & 3 \end{vmatrix} = \hat{\imath}(3+3) - \hat{\jmath}(3+1) + \hat{k}(3-1) = \langle 6, -4, 2 \rangle$$

To find a point of intersection, we pick $z = 0$ and solve:

$$\begin{matrix} x + y = 3 \\ x + 3y = 5 \end{matrix} \quad \Rightarrow \quad 2y = 2 \qquad y = 1 \qquad x = 2 \qquad P = (2, 1, 0)$$

The line is: $X = P + t\vec{v}$ $\quad (x,y,z) = (2,1,0) + t\langle 6, -4, 2 \rangle = (2 + 6t, 1 - 4t, 2t)$
Other answers are possible.