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MATH 221 Exam 2, Version A Spring 2024

501 Solutions P. Yasskin

1-9	/81	11	/12
10	/12	12	/105

Multiple Choice: (9 points each. No part credit.)
Circle you answers here and bubble on the Scantron.
Show your work, in case I give some part credit.

1. Find the plane tangent to the surface $z = xy^2 + x^3y$ at $(x,y) = (1,2)$.
What is the z-intercept?

a. $c = -14$ Correct
b. $c = -6$
c. $c = 0$
d. $c = 6$
e. $c = 14$

Solution: $f = xy^2 + x^3y$ $f_x = y^2 + 3x^2y$ $y_y = 2xy + x^3$ $f(1,2) = 6$ $f_x(1,2) = 10$ $y_y(1,2) = 5$

$z = f_{\text{tan}} = f(1,2) + f_x(1,2)(x-1) + y_y(1,2)(y-2) = 6 + 10(x-1) + 5(y-2)$

z-intercept occurs when $x = y = 0$. So $c = 6 + 10(-1) + 5(-2) = -14$

2. Find the plane tangent to the surface $xyz^2 = 6$ at $(x,y,z) = (3,2,1)$.
What is the z-intercept?

a. $c = -12$
b. $c = -2$
c. $c = 0$
d. $c = 2$ Correct
e. $c = 12$

Solution: $F = xyz^2$ $\vec{\nabla}F = \langle yz^2, xz^2, 2xyz\rangle$

$\vec{N} = \vec{\nabla}F\Big|_{(3.2.1)} = \langle 2,3,12\rangle$ $\vec{N} \cdot X = \vec{N} \cdot P$

$2x + 3y + 12z = 2(3) + 3(2) + 12(1) = 24$

z-intercept occurs when $x = y = 0$ and $z = c$. So $12c = 24$ $c = 2$

3. A weather balloon measures the temperature and its gradient at $P = (3,4,2)$ to be:

$$T = 70^\circ \qquad \vec{\nabla}T|_P = \langle -3,2,2 \rangle$$

Approximate the temperature at $(x,y,z) = (3.2,3.7,2.2)$.

- **a**. 68.4°
- **b**. 69.2° Correct
- **c**. 70.2°
- **d**. 70.8°
- **e**. 71.2°

Solution: $T(x,y,z) \approx T|_P + \left.\frac{\partial T}{\partial x}\right|_P (x-3) + \left.\frac{\partial T}{\partial y}\right|_P (y-4) + \left.\frac{\partial T}{\partial z}\right|_P (z-2)$

$T(3.2,3.7,2.2) \approx 70 - 3(3.2-3) + 2(3.7-4) + 2(2.2-2) = 70 - 3(.2) + 2(-.3) + 2(.2) = 69.2$

4. A weather balloon measures the temperature and its gradient at $P = (3,4,2)$ to be:

$$T = 70^\circ \qquad \vec{\nabla}T|_P = \langle -3,2,2 \rangle$$

If the balloon's velocity is $\vec{v} = \langle 2,4,-2 \rangle$, how fast is the temperature changing as seen aboard the balloon? $\frac{dT}{dt} =$

- **a**. -2 Correct
- **b**. -1
- **c**. 0
- **d**. 1
- **e**. 2

Solution: $\frac{dT}{dt} = \vec{v} \cdot \vec{\nabla}T|_P = \langle 2,4,-2 \rangle \cdot \langle -3,2,2 \rangle = -6 + 8 - 4 = -2$

5. The equation $x^2z + yz^2 = z^3 + 6$ implicitly defines z as a function of (x,y) near the point $P = (x,y,z) = (2,3,1)$. Find $\left.\frac{\partial z}{\partial y}\right|_P$.

- **a**. $\frac{\partial z}{\partial y} = \frac{-1}{13}$
- **b**. $\frac{\partial z}{\partial y} = \frac{-1}{\sqrt[3]{6}}$
- **c**. $\frac{\partial z}{\partial y} = \frac{-1}{7}$ Correct
- **d**. $\frac{\partial z}{\partial y} = \frac{1}{7}$
- **e**. $\frac{\partial z}{\partial y} = \frac{1}{13}$

Solution: We implicitly differentiate with respect to y. So x is constant.

$$x^2\frac{\partial z}{\partial y} + z^2 + y2z\frac{\partial z}{\partial y} = 3z^2\frac{\partial z}{\partial y}$$

We plug in numbers and simplify:

$$4\frac{\partial z}{\partial y} + 1 + 6\frac{\partial z}{\partial y} = 3\frac{\partial z}{\partial y} \qquad 7\frac{\partial z}{\partial y} = -1 \qquad \frac{\partial z}{\partial y} = \frac{-1}{7}$$

6. Two marbles are located at $P = (a,b)$ and $X = (x,y)$. Their current positions and velocities are:

$$P = (1,2) \qquad X = (5,5) \qquad \frac{dP}{dt} = \langle 15,-10 \rangle \qquad \frac{dX}{dt} = \langle 5,15 \rangle$$

How fast is the distance between them changing?
HINT: There are 4 intermediate variables.

a. $\frac{dD}{dt} = -1$
b. $\frac{dD}{dt} = 1$
c. $\frac{dD}{dt} = 3$
d. $\frac{dD}{dt} = 5$
e. $\frac{dD}{dt} = 7$ Correct

Solution: The distance between them is $D = \sqrt{(x-a)^2 + (y-b)^2}$. Currently, this is
$D = \sqrt{(5-1)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = 5$

The derivatives of D are

$$\frac{\partial D}{\partial a} = \frac{-(x-a)}{\sqrt{(x-a)^2 + (y-b)^2}} = \frac{-4}{5} \qquad \frac{\partial D}{\partial b} = \frac{-(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = \frac{-3}{5}$$

$$\frac{\partial D}{\partial x} = \frac{x-a}{\sqrt{(x-a)^2 + (y-b)^2}} = \frac{4}{5} \qquad \frac{\partial D}{\partial y} = \frac{y-b}{\sqrt{(x-a)^2 + (y-b)^2}} = \frac{3}{5}$$

The derivatives of a, b, c and d are given in the velocities. By the Chain Rule:

$$\frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial x}\frac{db}{dt} + \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} = \frac{-4}{5}(15) + \frac{-3}{5}(-10) + \frac{4}{5}(5) + \frac{3}{5}(15)$$

$$= -4(3) + -3(-2) + 4(1) + 3(3) = -12 + 6 + 4 + 9 = 7$$

7. Queen Lena is flying the Centurian Eagle through the Force whose density is $F = x^3y^2z \, \frac{\text{yodons}}{\text{lightsec}^3}$.
If she is located at $(x,y,z) = (1,2,3)$ and travels in the direction of maximum increase of the Force with speed $|\vec{v}| = 3\frac{\text{lightsec}}{\text{lightsec}}$, what is the rate she sees the Force increasing?

a. $\frac{dF}{dt} = 3\sqrt{91}$
b. $\frac{dF}{dt} = 4\sqrt{91}$
c. $\frac{dF}{dt} = 6\sqrt{91}$
d. $\frac{dF}{dt} = 12\sqrt{91}$ Correct
e. $\frac{dF}{dt} = 48\sqrt{91}$

Solution: The gradient of the force is $\vec{\nabla}F = \langle 3x^2y^2z, 2x^3yz, x^3y^2 \rangle = \langle 36,12,4 \rangle$.
The direction of maximum increase is $\hat{u} = \widehat{\nabla F}$ and the rate of change in this direction is
$\hat{u} \cdot \vec{\nabla}F = \left|\vec{\nabla}F\right| = \sqrt{36^2 + 12^2 + 4^2} = 4\sqrt{9^2 + 3^2 + 1^2} = 4\sqrt{91}$
Lena's velocity is $\vec{v} = |\vec{v}|\hat{u}$. So the rate she see the Force increasing is
$\frac{dF}{dt} = \vec{v} \cdot \vec{\nabla}F = |\vec{v}|\hat{u} \cdot \vec{\nabla}F = 3 \cdot 4\sqrt{91} = 12\sqrt{91}$

8. If $f(x,y) = x^2\cos(xy)$ which of the following is FALSE?

a. $f_{xx} = 2\cos(xy) - 4xy\sin(xy) - x^2y^2\cos(xy)$
b. $f_{xy} = -3x^2\sin(xy) - x^3y\cos(xy)$
c. $f_{yx} = -3x^2\cos xy + x^3y\sin xy$ Correct
d. $f_{yy} = -x^4\cos(xy)$

Solution: By Clairaut's Theorem, $f_{xy} = f_{yx}$. So (b) or (c) is false.
$f_x = 2x\cos(xy) - x^2y\sin(xy)$
$f_{xy} = -2x^2\sin(xy) - x^2\sin(xy) - x^3y\cos(xy) = -3x^2\sin(xy) - x^3y\cos(xy)$
So (c) must be false. We check:
$f_y = -x^3\sin(xy)$ $f_{yx} = -3x^2\sin(xy) - x^3y\cos(xy)$

9. The point $(2,-2)$ is a critical point of the function $f = y^3 - x^3 - 6xy$.
Classify this critical point using the Second Derivative Test.

a. Local Minimum
b. Local Maximum Correct
c. Inflection Point
d. Saddle Point
e. Test Fails

Solution: $f_x = -3x^2 - 6y$ $f_y = 3y^2 - 6x$ $f_x(2,-2) = -12 + 12 = 0$ $f_y(2,-2) = 12 - 12 = 0$
So $(2,-2)$ is a critical point.
$f_{xx} = -6x$ $f_{yy} = 6y$ $f_{xy} = -6$ $f_{xx}(2,-2) = -12$ $f_{yy}(2,-2) = -12$ $f_{xy}(2,-2) = -6$
$D = f_{xx}f_{yy} - f_{xy}^2 = 144 - 36 > 0$
Since $f_{xx} < 0$ and $D > 0$, the critical point is a Local Maximum.

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (12 pts) If the limit converges, prove it and find the limit.
If it diverges , give 2 curves which give different limits.

a. (6 pts) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

Solution: We try the straight lines: $y = mx$:

$$\lim_{\substack{y=mx\\x\to 0}} \frac{x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{xm}{x^2+m^2} = 0 \quad \text{for} \quad m \neq 0$$

We try the parabola: $y = x^2$:

$$\lim_{\substack{y=x^2\\x\to 0}} \frac{x^2x^2}{x^4+x^4} = \frac{1}{2} \neq 0$$ They are different. So the limit D.N.E.

b. (6 pts) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2}$

Solution: We try the straight lines: $y = mx$:

$$\lim_{\substack{y=mx\\x\to 0}} \frac{x^2m^3x^3}{(x^2+m^2x^2)^2} = \lim_{x\to 0} \frac{m^3x}{(1+m^2)^2} = 0$$

We try polar coordinates, $x = r\cos\theta,\, y = r\sin\theta$:

$$\lim_{\substack{x=r\cos\theta\\y=r\sin\theta}} \frac{r^2\cos^2\theta r^3\sin^3\theta}{r^4} = \lim_{r\to 0} r\cos^2\theta\sin^3\theta = 0$$ because $r \to 0$ and $\cos^2\theta\sin^3\theta$ is bounded.

11. (12 pts) Find the volume of the cylindrical can with the largest volume,
if its surface area is $A = 24\pi$.
HINT: The surface area is $A = 2\pi rh + 2\pi r^2$.

Solution: We maximize the volume $V = \pi r^2 h$ with the constraint $A = 2\pi rh + 2\pi r^2 = 24\pi$.

The gradients are $\vec{\nabla} V = \langle 2\pi rh, \pi r^2\rangle$ $\quad \vec{\nabla} A = \langle 2\pi h + 4\pi r, 2\pi r\rangle$

The Lagrange equations, $\vec{\nabla} V = \lambda\vec{\nabla} A$, are: $2\pi rh = \lambda(2\pi h + 4\pi r)$ $\quad \pi r^2 = \lambda 2\pi r$

We solve each for λ and equate: $\lambda = \dfrac{2\pi rh}{2\pi h + 4\pi r} = \dfrac{\pi r^2}{2\pi r}$

We cancel πr from the numerators and 2π from the denominators:

$\dfrac{2h}{h+2r} = \dfrac{r}{r} = 1$ So $2h = h + 2r$ $\quad h = 2r$

We substitute into the constraint: $A = 4\pi r^2 + 2\pi r^2 = 24\pi$ $\quad r^2 = 4$ $\quad r = 2$ $\quad h = 4$

So the volume is: $V = \pi r^2 h = \pi(2)^2 4 = 16\pi$