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MATH 221 Exam 3, Version A Spring 2024

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1-6	/54
7	/20
8	/15
9	/15
Total	/104

Multiple Choice: (9 points each. No part credit.)

Circle you answers here and bubble on the Scantron.

Show your work, in case I give some part credit.

1. Compute $\int_0^2 \int_p^{2p} pq\,dq\,dp$.

a. 4

b. 6 Correct

c. 8

d. 12

e. 24

Solution: $\int_0^2 \int_p^{2p} pq\,dq\,dp = \int_0^2 \left[p\frac{q^2}{2} \right]_{q=p}^{2p} dp = \int_0^2 \left(p\frac{4p^2}{2} \right) - \left(p\frac{p^2}{2} \right) dp = \frac{3}{2}\int_0^2 p^3\,dp = \frac{3}{2}\left[\frac{p^4}{4} \right]_0^2 = 6$

2. Approximate the integral $\int_2^6 \int_1^5 (x^2 + y)\,dy\,dx$ by a Riemann sum using 4 squares and evaluating at the center of each square.

a. 320 Correct

b. $\frac{976}{3}$

c. $\frac{488}{3}$

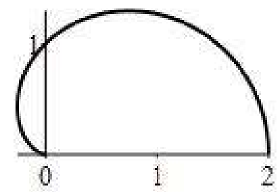
d. 160

e. 80

Solution: The squares have width and height $\Delta x = 2$ and $\Delta y = 2$ and so area $\Delta A = \Delta x \Delta y = 4$. The centers are $(3,2)$, $(3,4)$, $(5,2)$, $(5,4)$. The function is $f = x^2 + y$. The function values are: $f(3,2) = 11$, $f(3,4) = 13$, $f(5,2) = 27$, $f(5,4) = 29$ So the Riemann sum approximation is:

$$\int_2^6 \int_1^5 (x^2 + y)\,dy\,dx = \sum_{i=1}^{4} f(x_i, y_i)\Delta A = (11 + 13 + 27 + 29)4 = 320$$

3. Find the area of the interior of the upper half of the cardioid $r = 1 + \cos\theta$.



a. $A = \pi$

b. $A = 2\pi$

c. $A = \dfrac{2}{3\pi}$

d. $A = \dfrac{3\pi}{4}$ Correct

e. $A = \dfrac{3\pi}{2}$

Solution: In polar coordinates, $y = r\sin\theta$. The area is:

$$A = \int_0^\pi \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^\pi \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^\pi (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^\pi \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = \frac{3}{4}\pi$$

4. Find the average value of the function $f(x,y) = y$ on the interior of the upper half of the cardioid $r = 1 + \cos\theta$ as shown in problem 3.

a. $f_{\text{ave}} = \dfrac{3}{4}$

b. $f_{\text{ave}} = \dfrac{4}{3}$

c. $f_{\text{ave}} = \dfrac{4}{9\pi}$

d. $f_{\text{ave}} = \dfrac{9\pi}{16}$

e. $f_{\text{ave}} = \dfrac{16}{9\pi}$ Correct

Solution: The area was found in the previous problem to be $A = \frac{3}{4}\pi$.

In polar coordinates, $y = r\sin\theta$. So $f = y = r\sin\theta$. The integral of f is:

$$\iint f\,dA = \int_0^\pi \int_0^{1+\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^\pi \sin\theta\left[\frac{r^3}{3}\right]_0^{1+\cos\theta} d\theta = \frac{1}{3}\int_0^\pi (1+\cos\theta)^3 \sin\theta\,d\theta = -\frac{1}{3}\int u^3\,du$$

$$= -\frac{u^4}{12} = -\left[\frac{(1+\cos\theta)^4}{12}\right]_0^\pi = -\left(0 - \frac{2^4}{12}\right) = \frac{4}{3}$$

So the average is $f_{\text{ave}} = \dfrac{1}{A}\iint f\,dA = \dfrac{4}{3\pi}\dfrac{4}{3} = \dfrac{16}{9\pi}$

5. Find the volume of the apple given in spherical coordinates by $\rho = 1 - \cos\phi$.

a. $\frac{4\pi}{5}$
b. $\frac{8\pi}{5}$
c. $\frac{4\pi}{3}$
d. $\frac{8\pi}{3}$ Correct
e. $2\pi^2$

Solution: $V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi} \left[\frac{\rho^3}{3}\right]_0^{1-\cos\phi} \sin\phi\, d\phi$

$$= \frac{2\pi}{3}\int_0^{\pi}(1-\cos\phi)^3 \sin\phi\, d\phi = \frac{2\pi}{3}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} = \frac{\pi}{6}(2^4 - 0) = \frac{8\pi}{3}$$

6. Find the surface area of the parametric surface parametrized by $\vec{R}(u,v) = \langle u+v, u-v, 2u+2v\rangle$ for $0 \le u \le 2$ and $0 \le v \le 3$.

a. 36
b. 6
c. $2\sqrt{5}$
d. $6\sqrt{5}$
e. $12\sqrt{5}$ Correct

Solution:
$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = \langle & 1, & 1, & 2 \rangle \\ \vec{e}_\theta = \langle & 1, & -1, & 2 \rangle \end{matrix}$$
$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(2+2) - \hat{\jmath}(2-2) + \hat{k}(-1-1) = \langle 4, 0, -2\rangle$

$|\vec{N}| = \sqrt{16+4} = 2\sqrt{5}$ $\quad A = \iint 1\,dS = \iint |\vec{N}|\,du\,dv = \int_0^3\int_0^2 2\sqrt{5}\,du\,dv = 2\sqrt{5}\,(2)(3) = 12\sqrt{5}$

Work Out: (Points indicated. Part credit possible. Show all work.)

7. (20 points) Consider the region between the y-axis and the parabola $x = 9 - y^2$.

a. Find the mass of the region, if the surface density is $\delta = y^2$.

Solution: $M = \iint \delta\, dA = \int_{-3}^{3}\int_{0}^{9-y^2} y^2 dx\, dy = \int_{-3}^{3} y^2 \Big[x \Big]_0^{9-y^2} dy = \int_{-3}^{3} y^2(9-y^2)\, dy$

$= \int_{-3}^{3} (9y^2 - y^4)\, dy = \left[\frac{9y^3}{3} - \frac{y^5}{5} \right]_{-3}^{3} = 2\left(\frac{9 \cdot 3^3}{3} - \frac{3^5}{5} \right)$

$= 2 \cdot 3^5 \left(\frac{1}{3} - \frac{1}{5} \right) = 2 \cdot 3^5 \frac{2}{15} = \frac{4 \cdot 3^4}{5}$

b. Find the center of mass of the region, if the surface density is $\delta = y^2$.

Write your answer as an ordered pair.

Solution: From the previous part the mass is $M = \frac{4 \cdot 3^4}{5}$.

By symmetry $\bar{y} = 0$. The x-moment is

$M_x = \iint x\delta\, dA = \int_{-3}^{3}\int_{0}^{9-y^2} xy^2 dx\, dy = \int_{-3}^{3} y^2 \left[\frac{x^2}{2} \right]_0^{9-y^2} dy = \frac{1}{2}\int_{-3}^{3} y^2 (9-y^2)^2\, dy$

$= \frac{1}{2}\int_{-3}^{3} y^2(81 - 18y^2 + y^4)\, dy = \frac{1}{2}\int_{-3}^{3} (81y^2 - 18y^4 + y^6)\, dy = \frac{1}{2}\left[27y^3 - \frac{18y^5}{5} + \frac{y^7}{7} \right]_{-3}^{3}$

$= \left(27 \cdot 3^3 - \frac{18 \cdot 3^5}{5} + \frac{3^7}{7} \right) = 3^6\left(1 - \frac{6}{5} + \frac{3}{7} \right) = 3^6 \frac{35 - 42 + 15}{35} = \frac{8 \cdot 3^6}{35}$

$\bar{x} = \frac{M_x}{M} = \frac{8 \cdot 3^6}{35} \frac{5}{4 \cdot 3^4} = \frac{2 \cdot 9}{7} = \frac{18}{7}$

So the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{18}{7}, 0 \right)$.

8. (15 points) Compute $\iiint_P \vec{\nabla} \cdot \vec{F}\, dV$ for $\vec{F} = \langle xz, yz, z^2 \rangle$ over the solid above the paraboloid $z = x^2 + y^2$ below the plane $z = 4$.

Solution: $\vec{\nabla} \cdot \vec{F} = z + z + 2z = 4z$ bottom surface is $z = x^2 + y^2 = r^2$ $dV = r\, dr\, d\theta\, dz$

$\iiint_P \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 4z\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 \Big[2z^2 \Big]_{r^2}^4 r\, dr = 4\pi \int_0^2 (16 - r^4) r\, dr$

$= 4\pi \left[\frac{16r^2}{2} - \frac{r^6}{6} \right]_0^2 = 4\pi \left(\frac{64}{2} - \frac{64}{6} \right) = 256\pi \left(\frac{1}{2} - \frac{1}{6} \right) = \frac{256\pi}{3}$

9. (15 points) Compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = \langle -yz, xz, z^2 \rangle$

over the parabolic surface $z = x^2 + y^2$ below $z = 4$

oriented down and out. It may be parametrized by

$$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$$

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = \langle -x, -y, 2z \rangle$

On the surface $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}} = \langle -r\cos\theta, -r\sin\theta, 2r^2 \rangle$

$$\begin{array}{ll} \quad\ \ \hat{\imath} \quad\ \hat{\jmath} \quad\ \hat{k} & \\ \vec{e}_r = \Big\langle \cos\theta,\ \sin\theta,\ 2r \Big\rangle & \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) \\ \vec{e}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle & \quad = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle \end{array}$$

This is up and in. Need down and out.

Reverse $\vec{N} = \langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}} \cdot \vec{N} = -2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3 = -4r^3$$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_P \vec{\nabla} \times \vec{F}\Big|_{\vec{R}} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 -4r^3\, dr\, d\theta = -2\pi\Big[r^4\Big]_0^2 = -32\pi$$