MATH 221 Take Home Quiz 2 P. Yasskin

Version A Solutions

A) If your last name begins with A-F, consider the curve $\vec{r}(t) = (e^t \cos(t), e^t \sin(t), e^t)$.
Compute each of the following. Show your work. Simplify where possible.

1. velocity
$\vec{v}(t) = (e^t \cos(t) - e^t \sin(t), e^t \sin(t) + e^t \cos(t), e^t)$

2. acceleration
$\vec{a}(t) = (-2e^t \sin(t), 2e^t \cos(t), e^t)$

3. jerk
$\vec{j}(t) = (-2e^t \sin(t) - 2e^t \cos(t), 2e^t \cos(t) - 2e^t \sin(t), e^t)$

4. speed (HINT: The quantity in the square root is a perfect square.)
$$|\vec{v}(t)| = \sqrt{(e^t \cos(t) - e^t \sin(t))^2 + (e^t \sin(t) + e^t \cos(t))^2 + (e^t)^2}$$
$$= e^t \sqrt{2\cos^2(t) + 2\sin^2(t) + 1} = \sqrt{3}\, e^t$$

5. arclength between $t = 1$ and $t = 2$
$$L = \int_{(e\cos(1), e\sin(1), e)}^{(e^2\cos(2), e^2\sin(2), e^2)} ds = \int_1^2 |\vec{v}(t)|\, dt = \int_1^2 \sqrt{3}\, e^t\, dt = \left[\sqrt{3}\, e^t\right]_1^2 = \sqrt{3}\,(e^2 - e)$$

6. unit tangent vector
$$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \frac{1}{\sqrt{3}\, e^t}\vec{v} = \left(\frac{\cos(t) - \sin(t)}{\sqrt{3}}, \frac{\sin(t) + \cos(t)}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

7. $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^t \cos(t) - e^t \sin(t) & e^t \sin(t) + e^t \cos(t) & e^t \\ -2e^t \sin(t) & 2e^t \cos(t) & e^t \end{vmatrix}$$
$$= \hat{\imath}(e^{2t} \sin(t) + e^{2t} \cos(t) - 2e^{2t} \cos(t)) - \hat{\jmath}(e^{2t} \cos(t) - e^{2t} \sin(t) + 2e^{2t} \sin(t))$$
$$+ \hat{k}([e^t \cos(t) - e^t \sin(t)]2e^t \cos(t) + [e^t \sin(t) + e^t \cos(t)]2e^t \sin(t))$$
$$= (e^{2t} \sin(t) - e^{2t} \cos(t), -e^{2t} \cos(t) - e^{2t} \sin(t), 2e^{2t})$$

8. $$|\vec{v} \times \vec{a}| = \sqrt{(e^{2t} \sin(t) - e^{2t} \cos(t))^2 + (-e^{2t} \cos(t) - e^{2t} \sin(t))^2 + (2e^{2t})^2}$$
$$= e^{2t} \sqrt{2\sin^2(t) + 2\cos^2(t) + 4} = \sqrt{6}\, e^{2t}$$

9. unit binormal vector
$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{\sqrt{6}\, e^{2t}}(e^{2t} \sin(t) - e^{2t} \cos(t), -e^{2t} \cos(t) - e^{2t} \sin(t), 2e^{2t})$$
$$= \left(\frac{\sin(t) - \cos(t)}{\sqrt{6}}, \frac{-\cos(t) - \sin(t)}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{3}\sqrt{6}} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin(t)-\cos(t) & -\cos(t)-\sin(t) & 2 \\ \cos(t)-\sin(t) & \sin(t)+\cos(t) & 1 \end{vmatrix}$$

$$= \frac{1}{3\sqrt{2}} \left[\begin{array}{c} \hat{\imath}(-\cos(t)-\sin(t)-2(\sin(t)+\cos(t))) - \hat{\jmath}(\sin(t)-\cos(t)-2(\cos(t)-\sin(t))) \\ +\hat{k}((\sin(t)-\cos(t))(\sin(t)+\cos(t)) - (-\cos(t)-\sin(t))(\cos(t)-\sin(t))) \end{array} \right]$$

$$= \frac{1}{3\sqrt{2}}(-3(\sin(t)+\cos(t)), 3(\cos(t)-\sin(t)), 0) = \left(-\frac{1}{\sqrt{2}}(\sin(t)+\cos(t)), \frac{1}{\sqrt{2}}(\cos(t)-\sin(t)), 0\right)$$

11. curvature

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{6}\, e^{2t}}{\left(\sqrt{3}\, e^t\right)^3} = \frac{\sqrt{2}}{3 e^t}$$

12. torsion

$$\tau = \frac{\vec{v} \times \vec{a} \bullet \vec{j}}{|\vec{v} \times \vec{a}|^2} = \left(\frac{1}{\sqrt{6}\, e^{2t}}\right)^2 (e^{2t}\sin(t) - e^{2t}\cos(t), -e^{2t}\cos(t) - e^{2t}\sin(t), 2e^{2t})$$

$$\bullet\, (-2e^t\sin(t) - 2e^t\cos(t), 2e^t\cos(t) - 2e^t\sin(t), e^t)$$

$$= \frac{1}{6e^{4t}} e^{3t} 2 = \frac{1}{3e^t}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a} \bullet \hat{T} = (-2e^t\sin(t), 2e^t\cos(t), e^t) \bullet \frac{1}{\sqrt{3}}(\cos(t)-\sin(t), \sin(t)+\cos(t), 1) = \sqrt{3}\, e^t$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\sqrt{3}\, e^t\right) = \sqrt{3}\, e^t$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a} \bullet \hat{N} = (-2e^t\sin(t), 2e^t\cos(t), e^t) \bullet \frac{1}{\sqrt{2}}(-(\sin(t)+\cos(t)), (\cos(t)-\sin(t)), 0) = \sqrt{2}\, e^t$$

$$a_N = \kappa|\vec{v}|^2 = \frac{\sqrt{2}}{3e^t}\left(\sqrt{3}\, e^t\right)^2 = \sqrt{2}\, e^t$$

15. mass of a wire between $t = 1$ and $t = 2$ with linear density $\rho = x$

$$M = \int_{(e\cos(1), e\sin(1), e)}^{(e^2\cos(2), e^2\sin(2), e^2)} \rho\, ds = \int_1^2 \rho(r(t))|\vec{v}(t)|\, dt = \int_1^2 e^t\cos(t)\, \sqrt{3}\, e^t\, dt$$

Using integration by parts, $\int e^{2t}\cos(t)\, dt = \frac{1}{5}e^{2t}(2\cos t + \sin t)$ So

$$M = \left[\frac{1}{5}e^{2t}(2\cos t + \sin t)\right]_1^2 = \frac{1}{5}e^4(2\cos 2 + \sin 2) - \frac{1}{5}e^2(2\cos 1 + \sin 1)$$

16. work to move a bead along the wire from $t = 1$ to $t = 2$.

For curves A and E, the force is $\vec{F} = (-y, x, 0)$.

$\vec{F} = (-y, x, 0) \qquad \vec{F}(\vec{r}(t)) = (-e^t\sin(t), e^t\cos(t), 0)$

$$W = \int_{(e\cos(1), e\sin(1), e)}^{(e^2\cos(2), e^2\sin(2), e^2)} \vec{F} \bullet d\vec{s} = \int_1^2 \vec{F}(r(t)) \bullet \vec{v}(t)\, dt$$

$$= \int_1^2 (-e^t\sin(t), e^t\cos(t), 0) \bullet (e^t\cos(t) - e^t\sin(t), e^t\sin(t) + e^t\cos(t), e^t)\, dt$$

$$= \int_1^2 e^{2t}\, dt = \left[\frac{e^{2t}}{2}\right]_1^2 = \frac{1}{2}(e^4 - e^2)$$