MATH 221 Take Home Quiz 2 P. Yasskin

Version B Solutions

B) If your last name begins with G-L, consider the curve $\vec{r}(t) = (3t^2, 4t^3, 3t^4)$.
Compute each of the following. Show your work. Simplify where possible.

1. velocity
$\vec{v}(t) = (6t, 12t^2, 12t^3)$

2. acceleration
$\vec{a}(t) = (6, 24t, 36t^2)$

3. jerk
$\vec{j}(t) = (0, 24, 72t)$

4. speed (HINT: The quantity in the square root is a perfect square.)
$|\vec{v}(t)| = \sqrt{36t^2 + 144t^4 + 144t^6} = 6t\sqrt{1 + 4t^2 + 4t^4} = 6t(1 + 2t^2) = 6t + 12t^3$

5. arclength between $t = 1$ and $t = 2$
$$L = \int_{(3,4,3)}^{(12,32,48)} ds = \int_1^2 |\vec{v}(t)|\,dt = \int_1^2 (6t + 12t^3)\,dt = \left[3t^2 + 3t^4\right]_1^2 = (12 + 48) - (3 + 3) = 54$$

6. unit tangent vector
$$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \frac{1}{6t + 12t^3}(6t, 12t^2, 12t^3) = \frac{1}{1 + 2t^2}(1, 2t, 2t^2) = \left(\frac{1}{1 + 2t^2}, \frac{2t}{1 + 2t^2}, \frac{2t^2}{1 + 2t^2}\right)$$

7. $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6t & 12t^2 & 12t^3 \\ 6 & 24t & 36t^2 \end{vmatrix} = 36t\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 2t^2 \\ 1 & 4t & 6t^2 \end{vmatrix} = 36t\left[\hat{\imath}(12t^3 - 8t^3) - \hat{\jmath}(6t^2 - 2t^2) + \hat{k}(4t - 2t)\right]$$
$$= 36t(4t^3, -4t^2, 2t) = 72t^2(2t^2, -2t, 1)$$

8. $|\vec{v} \times \vec{a}| = 72t^2\sqrt{4t^4 + 4t^2 + 1} = 72t^2(1 + 2t^2)$

9. unit binormal vector
$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{72t^2(1 + 2t^2)} 72t^2(2t^2, -2t, 1) = \left(\frac{2t^2}{1 + 2t^2}, \frac{-2t}{1 + 2t^2}, \frac{1}{1 + 2t^2}\right)$$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{(1+2t^2)^2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t^2 & -2t & 1 \\ 1 & 2t & 2t^2 \end{vmatrix} = \frac{1}{(1+2t^2)^2}\left[\hat{\imath}(-4t^3-2t) - \hat{\jmath}(4t^4-1) + \hat{k}(4t^3+2t)\right]$$

$$= \frac{1}{(1+2t^2)^2}(-2t(2t^2+1),(1-2t^2)(1+2t^2),2t(2t^2+1)) = \left(\frac{-2t}{1+2t^2},\frac{1-2t^2}{1+2t^2},\frac{2t}{1+2t^2}\right)$$

11. curvature

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \left(\frac{1}{6t(1+2t^2)}\right)^3 72t^2(1+2t^2) = \frac{1}{3t(1+2t^2)^2}$$

12. torsion

$$\tau = \frac{\vec{v}\times\vec{a}\cdot\vec{j}}{|\vec{v}\times\vec{a}|^2} = \left(\frac{1}{72t^2(1+2t^2)}\right)^2 72t^2(2t^2,-2t,1)\cdot(0,24,72t) = \frac{1}{72t^2(1+2t^2)^2}(-48t+72t)$$

$$= \frac{1}{3t(2t^2+1)^2}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a}\cdot\hat{T} = (6,24t,36t^2)\cdot\frac{1}{1+2t^2}(1,2t,2t^2) = \frac{1}{1+2t^2}(6+48t^2+72t^4) = 6+36t^2$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(6t+12t^3) = 6+36t^2$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a}\cdot\hat{N} = (6,24t,36t^2)\cdot\frac{1}{1+2t^2}(-2t,1-2t^2,2t) = \frac{1}{1+2t^2}(-12t+24t-48t^3+72t^3)$$

$$= \frac{12t+24t^3}{1+2t^2} = 12t$$

$$a_N = \kappa|\vec{v}|^2 = \frac{1}{3t(1+2t^2)^2}(6t(1+2t^2))^2 = 12t$$

15. mass of a wire between $t=1$ and $t=2$ with linear density $\rho = x$

$$M = \int_{(3,4,3)}^{(12,32,48)}\rho\,ds = \int_1^2 \rho(r(t))|\vec{v}(t)|\,dt = \int_1^2 3t^2(6t+12t^3)\,dt = \int_1^2(18t^3+36t^5)\,dt$$

$$= \left[\frac{18t^4}{4}+\frac{36t^6}{6}\right]_1^2 = (72+384)-\left(\frac{9}{2}+6\right) = \frac{891}{2}$$

16. work to move a bead along the wire from $t=1$ to $t=2$.
For curves B, C and D, the force is $\vec{F} = (0,y,x)$.

$$\vec{F} = (0,y,x) \qquad \vec{F}(\vec{r}(t)) = (0,4t^3,3t^2)$$

$$W = \int_{(3,4,3)}^{(12,32,48)}\vec{F}\cdot d\vec{s} = \int_1^2 \vec{F}(r(t))\cdot\vec{v}(t)\,dt = \int_1^2(0,4t^3,3t^2)\cdot(6t,12t^2,12t^3)\,dt$$

$$= \int_1^2(48t^5+36t^5)\,dt = \int_1^2 84t^5\,dt = \left[\frac{84t^6}{6}\right]_1^2 = 882$$