MATH 221 Take Home Quiz 2 P. Yasskin

Version C Solutions

C) If your last name begins with M-R, consider the curve $\vec{r}(t) = \left(e^t, \sqrt{2}\,t, e^{-t}\right)$.
Compute each of the following. Show your work. Simplify where possible.

1. velocity
$\vec{v}(t) = \left(e^t, \sqrt{2}, -e^{-t}\right)$

2. acceleration
$\vec{a}(t) = (e^t, 0, e^{-t})$

3. jerk
$\vec{j}(t) = (e^t, 0, -e^{-t})$

4. speed (HINT: The quantity in the square root is a perfect square.)
$|\vec{v}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

5. arclength between $t = 1$ and $t = 2$

$$L = \int_{\left(e,\sqrt{2},1/e\right)}^{\left(e^2,2\sqrt{2},1/e^2\right)} ds = \int_1^2 |\vec{v}(t)|\,dt = \int_1^2 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_1^2$$
$$= \left(e^2 - \frac{1}{e^2}\right) - \left(e - \frac{1}{e}\right) = e^2 - e + \frac{1}{e} - \frac{1}{e^2}$$

6. unit tangent vector

$$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \left(\frac{e^t}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}}\right)$$

7. $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^t & \sqrt{2} & -e^{-t} \\ e^t & 0 & e^{-t} \end{vmatrix} = \hat{\imath}\left(\sqrt{2}\,e^{-t}\right) - \hat{\jmath}(1 + 1) + \hat{k}\left(-\sqrt{2}\,e^t\right) = \left(\sqrt{2}\,e^{-t}, -2, -\sqrt{2}\,e^t\right)$$

8. $|\vec{v} \times \vec{a}| = \sqrt{2e^{-2t} + 4 + 2e^{2t}} = \sqrt{2}\,(e^t + e^{-t})$

9. unit binormal vector

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{\sqrt{2}\,(e^t + e^{-t})}\left(\sqrt{2}\,e^{-t}, -2, -\sqrt{2}\,e^t\right) = \left(\frac{e^{-t}}{e^t + e^{-t}}, \frac{-\sqrt{2}}{e^t + e^{-t}}, \frac{-e^t}{e^t + e^{-t}}\right)$$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{(e^t + e^{-t})^2} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^{-t} & -\sqrt{2} & -e^t \\ e^t & \sqrt{2} & -e^{-t} \end{vmatrix}$$

$$= \frac{1}{(e^t + e^{-t})^2} \left[\hat{\imath}\left(\sqrt{2}\, e^{-t} + \sqrt{2}\, e^t\right) - \hat{\jmath}(-e^{-2t} + e^{2t}) + \hat{k}\left(\sqrt{2}\, e^{-t} + \sqrt{2}\, e^t\right) \right]$$

$$= \frac{1}{(e^t + e^{-t})^2} \left(\sqrt{2}\,(e^{-t} + e^t), (e^{-t} + e^t)(e^{-t} - e^t), \sqrt{2}\,(e^{-t} + e^t)\right) = \left(\frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^{-t} - e^t}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}} \right)$$

11. curvature

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{2}\,(e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

12. torsion

$$\tau = \frac{\vec{v} \times \vec{a} \cdot \vec{j}}{|\vec{v} \times \vec{a}|^2} = \frac{\left(\sqrt{2}\, e^{-t}, -2, -\sqrt{2}\, e^t\right) \cdot (e^t, 0, -e^{-t})}{\left[\sqrt{2}\,(e^t + e^{-t})\right]^2} = \frac{\sqrt{2} + 0 + \sqrt{2}}{2(e^t + e^{-t})^2} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a} \cdot \hat{T} = (e^t, 0, e^{-t}) \cdot \left(\frac{e^t}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}} \right) = \frac{e^{2t}}{e^t + e^{-t}} + \frac{-e^{-2t}}{e^t + e^{-t}}$$

$$= \frac{(e^t + e^{-t})(e^t - e^{-t})}{e^t + e^{-t}} = e^t - e^{-t}$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t - e^{-t}$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a} \cdot \hat{N} = (e^t, 0, e^{-t}) \cdot \left(\frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^{-t} - e^t}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}} \right) = \frac{\sqrt{2}\, e^t}{e^t + e^{-t}} + \frac{\sqrt{2}\, e^{-t}}{e^t + e^{-t}} = \sqrt{2}$$

$$a_N = \kappa |\vec{v}|^2 = \frac{\sqrt{2}}{(e^t + e^{-t})^2}(e^t + e^{-t})^2 = \sqrt{2}$$

15. mass of a wire between $t = 1$ to $t = 2$ with linear density $\rho = x$

$$M = \int_{(e,\sqrt{2},1/e)}^{(e^2,2\sqrt{2},1/e^2)} \rho\, ds = \int_1^2 \rho(r(t))|\vec{v}(t)|\, dt = \int_1^2 (e^t)(e^t + e^{-t})\, dt = \int_1^2 (e^{2t} + 1)\, dt$$

$$= \left[\frac{e^{2t}}{2} + t \right]_1^2 = \left(\frac{1}{2} e^4 + 2 \right) - \left(\frac{1}{2} e^2 + 1 \right) = \frac{1}{2}(e^4 - e^2 + 2)$$

16. work to move a bead along the wire from $t = 1$ to $t = 2$.
For curves B, C and D, the force is $\vec{F} = (0, y, x)$.

$$\vec{F} = (0, y, x) \qquad \vec{F}(\vec{r}(t)) = \left(0, \sqrt{2}\, t, e^t\right)$$

$$W = \int_{(e,\sqrt{2},1/e)}^{(e^2,2\sqrt{2},1/e^2)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F}(r(t)) \cdot \vec{v}(t)\, dt = \int_1^2 \left(0, \sqrt{2}\, t, e^t\right) \cdot \left(e^t, \sqrt{2}, -e^{-t}\right) dt$$

$$= \int_1^2 (2t - 1)\, dt = [t^2 - t]_1^2 = (4 - 2) - (1 - 1) = 2$$