MATH 221 Take Home Quiz 2 P. Yasskin

Sample Solutions

Consider the sample curve $\vec{r}(t) = \left(2t\cos(t), 2t\sin(t), \frac{1}{3}t^3\right)$.

Compute each of the following. Show your work. Simplify where possible.

1. velocity

$$\vec{v}(t) = (2\cos(t) - 2t\sin(t), 2\sin(t) + 2t\cos(t), t^2)$$

2. acceleration

$$\vec{a}(t) = (-4\sin(t) - 2t\cos(t), 4\cos(t) - 2t\sin(t), 2t)$$

3. jerk

$$\vec{j}(t) = (-6\cos(t) + 2t\sin(t), -6\sin(t) - 2t\cos(t), 2)$$

4. speed (HINT: The quantity in the square root is a perfect square.)

$$|\vec{v}(t)| = \sqrt{[2\cos(t) - 2t\sin(t)]^2 + [2\sin(t) + 2t\cos(t)]^2 + [t^2]^2} = \sqrt{4 + 4t^2 + t^4} = 2 + t^2$$

5. arclength between $t = 0$ and $t = 1$

$$L = \int_{(0,0,0)}^{\left(2\cos(1), 2\sin(1), \frac{1}{3}\right)} ds = \int_0^1 |\vec{v}(t)|\,dt = \int_0^1 (2 + t^2)\,dt = \left[2t + \frac{1}{3}t^3\right]_0^1 = \frac{7}{3}$$

6. unit tangent vector

$$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \frac{1}{2 + t^2}\vec{v} = \left(\frac{2\cos(t) - 2t\sin(t)}{2 + t^2}, \frac{2\sin(t) + 2t\cos(t)}{2 + t^2}, \frac{t^2}{2 + t^2}\right)$$

7. $\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos(t) - 2t\sin(t) & 2\sin(t) + 2t\cos(t) & t^2 \\ -4\sin(t) - 2t\cos(t) & 4\cos(t) - 2t\sin(t) & 2t \end{vmatrix}$

$$= \hat{\imath}(2t(2 + t^2)\sin(t)) - \hat{\jmath}(2t(2 + t^2)\cos(t)) + \hat{k}(4t^2 + 8)$$

$$= (2t(2 + t^2)\sin(t), -2t(2 + t^2)\cos(t), 4(2 + t^2))$$

8. $|\vec{v} \times \vec{a}| = \sqrt{[2t(2 + t^2)\sin(t)]^2 + [-2t(2 + t^2)\cos(t)]^2 + [4(2 + t^2)]^2}$

$$= (2 + t^2)\sqrt{[2t\sin(t)]^2 + [-2t\cos(t)]^2 + [4]^2} = (2 + t^2)\sqrt{4t^2 + 16} = 2(2 + t^2)\sqrt{t^2 + 4}$$

9. unit binormal vector

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{2(2 + t^2)\sqrt{t^2 + 4}}(2t(2 + t^2)\sin(t), -2t(2 + t^2)\cos(t), 4(2 + t^2))$$

$$= \left(\frac{t\sin(t)}{\sqrt{t^2 + 4}}, \frac{-t\cos(t)}{\sqrt{t^2 + 4}}, \frac{2}{\sqrt{t^2 + 4}}\right)$$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{(2+t^2)\sqrt{t^2+4}} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t\sin(t) & -t\cos(t) & 2 \\ 2\cos(t) - 2t\sin(t) & 2\sin(t) + 2t\cos(t) & t^2 \end{vmatrix}$$

$$= \frac{1}{(2+t^2)\sqrt{t^2+4}} \left[\begin{array}{c} \hat{\imath}(-t^3\cos(t) - 2(2\sin(t) + 2t\cos(t))) - \hat{\jmath}(t^3\sin(t) - 2(2\cos(t) - 2t\sin(t))) \\ +\hat{k}(t\sin(t)(2\sin(t) + 2t\cos(t)) + t\cos(t)(2\cos(t) - 2t\sin(t))) \end{array} \right]$$

$$= \frac{1}{(2+t^2)\sqrt{t^2+4}} (-4\sin(t) - (4t+t^3)\cos(t), 4\cos(t) - (4t+t^3)\sin(t), 2t)$$

11. curvature

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{2(2+t^2)\sqrt{t^2+4}}{(2+t^2)^3} = \frac{2\sqrt{t^2+4}}{(2+t^2)^2}$$

12. torsion

$$\vec{v} \times \vec{a} \cdot \vec{j} = (2t(2+t^2)\sin(t), -2t(2+t^2)\cos(t), 4(2+t^2)) \cdot (-6\cos(t) + 2t\sin(t), -6\sin(t) - 2t\cos(t), 2)$$

$$= 4(2+t^2)^2$$

$$|\vec{v} \times \vec{a}|^2 = \left[2(2+t^2)\sqrt{t^2+4}\,\right]^2 = 4(2+t^2)^2(t^2+4)$$

$$\tau = \frac{\vec{v} \times \vec{a} \cdot \vec{j}}{|\vec{v} \times \vec{a}|^2} = \frac{4(2+t^2)^2}{4(2+t^2)^2(t^2+4)} = \frac{1}{t^2+4}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a} \cdot \hat{T} = (-4\sin(t) - 2t\cos(t), 4\cos(t) - 2t\sin(t), 2t) \cdot \frac{1}{2+t^2}(2\cos(t) - 2t\sin(t), 2\sin(t) + 2t\cos(t), t^2)$$

$$= \frac{2t^3 + 4t}{2+t^2} = 2t$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(2+t^2) = 2t$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a} \cdot \hat{N} = (-4\sin(t) - 2t\cos(t), 4\cos(t) - 2t\sin(t), 2t)$$

$$\cdot \frac{1}{(2+t^2)\sqrt{t^2+4}}(-4\sin(t) - (4t+t^3)\cos(t), 4\cos(t) - (4t+t^3)\sin(t), 2t)$$

$$= \frac{2(2+t^2)(t^2+4)}{(2+t^2)\sqrt{t^2+4}} = 2\sqrt{t^2+4}$$

$$a_N = \kappa|\vec{v}|^2 = \frac{2\sqrt{t^2+4}}{(2+t^2)^2}(2+t^2)^2 = 2\sqrt{t^2+4}$$

15. mass of a wire between $t = 0$ and $t = 1$ with linear density $\rho = x^2 + y^2$

$$M = \int_{(0,0,0)}^{\left(2\cos(1), 2\sin(1), \frac{1}{3}\right)} \rho\, ds = \int_0^1 \rho(r(t))|\vec{v}(t)|\, dt = \int_0^1 (4t^2)(2+t^2)\, dt = \left[8\frac{t^3}{3} + 4\frac{t^5}{5}\right]_0^1 = \frac{8}{3} + \frac{4}{5} = \frac{52}{15}$$

16. work to move a bead along the wire from $t = 0$ to $t = 1$ with the force $\vec{F} = (-y, x, 0)$

$$\vec{F}(\vec{r}(t)) = (-2t\sin(t), 2t\cos(t), 0) \qquad \vec{v}(t) = (2\cos(t) - 2t\sin(t), 2\sin(t) + 2t\cos(t), t^2)$$

$$W = \int_{(0,0,0)}^{\left(2\cos(1), 2\sin(1), \frac{1}{3}\right)} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(r(t)) \cdot \vec{v}(t)\, dt = \int_0^1 4t^2\, dt = \left[4\frac{t^3}{3}\right]_0^1 = \frac{4}{3}$$