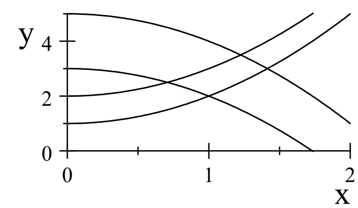
MATH 221 Take Home Quiz 3 P. Yasskin

Solutions

1. Compute the integral $\iint x\,dA$ over the region in the first quadrant bounded by $y = 1 + x^2$, $y = 2 + x^2$, $y = 3 - x^2$, and $y = 5 - x^2$.



a. Define the curvilinear coordinates u and v by $y = u + x^2$ and $y = v - x^2$.
What are the 4 boundaries in terms of u and v?

$u = 1 \qquad u = 2 \qquad v = 3 \qquad v = 5$

b. Solve for x and y in terms of u and v. Express the results as a position vector.

Add and subtract: $2y = u + x^2 + v - x^2 = u + v \qquad y = \dfrac{u+v}{2}$

$y - y = u + x^2 - v + x^2 = u - v + 2x^2 \qquad 2x^2 = v - u \qquad x = \dfrac{\sqrt{v-u}}{\sqrt{2}}$

$$\vec{R}(u,v) = (x(u,v), y(u,v)) = \left(\frac{\sqrt{v-u}}{\sqrt{2}}, \frac{u+v}{2} \right)$$

c. Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \left(\frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}}, \frac{1}{2} \right)$$

$$\vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \left(\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}, \frac{1}{2} \right)$$

d. Compute the Jacobian determinant:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}} \frac{-1}{\sqrt{v-u}} - \frac{1}{4\sqrt{2}} \frac{1}{\sqrt{v-u}} = \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}}$$

e. Compute the Jacobian factor:

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}$$

f. Compute the integral:

$$\iint x\,dA = \int_3^4 \int_1^2 \frac{\sqrt{v-u}}{\sqrt{2}} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}\,du\,dv = \int_3^5 \int_1^2 \frac{1}{4}\,du\,dv = \frac{1}{4}(5-3)(2-1) = \frac{1}{2}$$

2. Find the Jacobian for spherical coordinates. The position vector is given by

$$\vec{R}(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

a. Find the coordinate tangent vectors:

$$\vec{e}_\rho = \frac{\partial\vec{R}}{\partial\rho} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$\vec{e}_\theta = \frac{\partial\vec{R}}{\partial\theta} = (-\rho\sin\varphi\sin\theta, \rho\sin\varphi\cos\theta, 0)$$

$$\vec{e}_\varphi = \frac{\partial\vec{R}}{\partial\varphi} = (\rho\cos\varphi\cos\theta, \rho\cos\varphi\sin\theta, -\rho\sin\varphi)$$

b. Compute the Jacobian determinant:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \end{vmatrix}$$

Expand on the third column:

$$= \cos\varphi(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta) - \rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta - \rho\sin^2\varphi\sin^2\theta)$$

$$= \cos\varphi(-\rho^2\sin\varphi\cos\varphi) - \rho\sin\varphi(\rho\sin^2\varphi)$$

$$= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi = -\rho^2\sin\varphi$$

$$= -\rho^2\sin\varphi$$

c. Compute the Jacobian factor:

$$J = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\sin\varphi$$