MATH 151 Honors, FALL SEMESTER 2011
FINAL EXAMINATION - SOLUTIONS

Name (print): ____________________ Instructor's name: Yasskin

Signature: ____________________ Section No: ________

Part 1 – Multiple Choice (15 questions, 4 points each, No Calculators)

Write your name and section number on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. Evaluate $\lim\limits_{x\to-2} \dfrac{x^3-x^2-6x}{x^2-4}$

a. 3
b. ∞
c. 0
d. $-\dfrac{5}{2}$ Correct Choice
e. 6

SOLUTION: $\lim\limits_{x\to-2} \dfrac{x^3-x^2-6x}{x^2-4} = \lim\limits_{x\to-2} \dfrac{x(x+2)(x-3)}{(x+2)(x-2)} = \lim\limits_{x\to-2} \dfrac{x(x-3)}{(x-2)} = \dfrac{(-2)(-2-3)}{(-2-2)} = -\dfrac{5}{2}$

2. The limit $\lim\limits_{h\to0} \dfrac{4(2+h)^3-32}{h}$ can be interpreted as which of the following?

a. $f'(32)$ where $f(x) = 4x^3$
b. $f'(2)$ where $f(x) = 4x^3$ Correct Choice
c. $f'(4)$ where $f(x) = x^3$
d. $f'(2)$ where $f(x) = 12x^2$
e. $f'(2)$ where $f(x) = x^4$

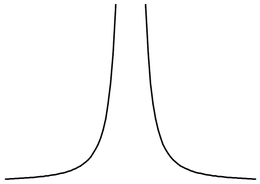
SOLUTION: $f(x+h) = 4(2+h)^3$ So $x = 2$ and $f(x) = 4x^3$ and $f(2) = 32$.

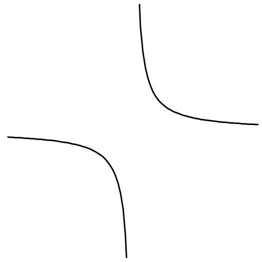
3. Find the line tangent to $y = \sinh x$ at $x = 1$. Its y-intercept is

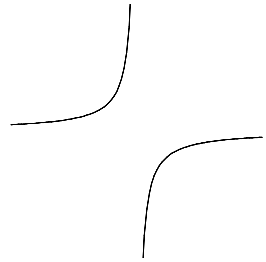
a. 0
b. $\dfrac{e}{2}$
c. e
d. $-\dfrac{1}{e}$ Correct Choice
e. $-\dfrac{1}{2e}$

SOLUTION: $f(x) = \sinh x$ $f'(x) = \cosh x$ $f(1) = \sinh 1$ $f'(1) = \cosh 1$
Tan Line: $y = f(1) + f'(1)(x-1) = \sinh 1 + \cosh 1(x-1)$
$b = y(0) = \sinh 1 + \cosh 1(-1) = \dfrac{e^1-e^{-1}}{2} - \dfrac{e^1+e^{-1}}{2} = -e^{-1}$

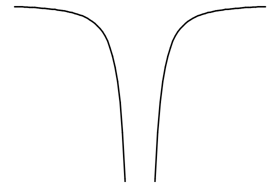
4. The function $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4x + 4}$ has a vertical asymptote at $x = 2$. Near $x = 2$, the graph has the shape:

a. 

b. 

c. 

Correct Choice

d. 

SOLUTION: $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4x + 4} = \dfrac{(x-2)(x-3)}{(x-2)^2} = \dfrac{(x-3)}{(x-2)}$

$\lim\limits_{x \to 2^-} \dfrac{(x-3)}{(x-2)} = \dfrac{\text{“}-1^{-}\text{”}}{0^-} = +\infty \qquad \lim\limits_{x \to 2^+} \dfrac{(x-3)}{(x-2)} = \dfrac{\text{“}-1^{+}\text{”}}{0^+} = -\infty$

5. Find the absolute minimum value of $f(x) = \sin\left(x - \frac{\pi}{6}\right) - \frac{x}{2}$ on the interval $[0, \pi]$.

NOTE: $\dfrac{\sqrt{3}}{2} \approx 0.806 \qquad \dfrac{\pi}{4} \approx 0.785 \qquad \dfrac{\pi}{2} \approx 1.570$

a. $-\dfrac{1}{2}$

b. $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{4}$

c. $\dfrac{1}{2} - \dfrac{\pi}{2}$ Correct Choice

d. $-\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{4}$

e. $-\dfrac{1}{2} - \dfrac{\pi}{2}$

SOLUTION: $f'(x) = \cos\left(x - \frac{\pi}{6}\right) - \frac{1}{2} = 0 \qquad x - \frac{\pi}{6} = \frac{\pi}{3} \qquad x = \frac{\pi}{2}$

Check critical points in the interval and endpoints:

$f(0) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - \frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\pi}{4} \approx 0.806 - 0.785 > 0$

$f(\pi) = \sin\left(\frac{5\pi}{6}\right) - \frac{\pi}{2} = \frac{1}{2} - \frac{\pi}{2} \approx .5 - 1.570 \approx -1.07 \qquad \Leftarrow$ absolute minimum

6. If $\$1000$ is invested at 2% annual interest compounded continuously, how long will it take for the principal to reach $\$1000e \approx \$2718.28$?

a. 5 years

b. 10 years

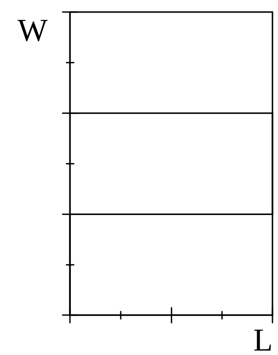
c. 25 years

d. 50 years Correct Choice

e. 100 years

SOLUTION: $P = P_0 e^{rt} \qquad 1000e = 1000e^{0.02t} \qquad e = e^{0.02t} \qquad 1 = 0.02t \qquad t = 50$

7. A rectangular field is surrounded by a fence and divided into 3 pens by 2 additional fences parallel to one side. If the total area is 72 m^2 find the minimum total length of fence.



a. 48 Correct Choice

b. 50

c. 52

d. 60

e. 66

SOLUTION: $A = LW = 72$ $\quad F = 4L + 2W = 4L + \frac{144}{L}$ $\quad F' = 4 - \frac{144}{L^2} = 0$

$4 = \frac{144}{L^2}$ $\quad L^2 = 36$ $\quad L = 6$ $\quad W = \frac{72}{6} = 12$ $\quad F = 4 \cdot 6 + 2 \cdot 12 = 48$

8. Let $f(x)$ be a differentiable function, and suppose $f(1) = -5$ and $f'(x) \ge -7$ for all values of x. Use the Mean Value Theorem to determine how small $f(4)$ can possibly be.

a. 26

b. -26 Correct Choice

c. -20

d. 35

e. Not enough information.

SOLUTION: By the MVT, there is a c in $(1,4)$ where

$\frac{f(4) - f(1)}{4 - 1} = f'(c) \ge -7$ $\quad$ So $f(4) \ge f(1) - 7(4 - 1) = -5 - 7(3) = -26$

9. When you prove $\lim_{x \to 4}(3x - 5) = 7$, and pick $\varepsilon > 0$, the largest possible δ is

a. ε

b. $\frac{\varepsilon}{3}$ Correct Choice

c. $\frac{\varepsilon}{2}$

d. 2ε

e. 3ε

SOLUTION: $\lim_{x \to 4}(3x - 5) = 7$ means

For all $\varepsilon > 0$ there is a $\delta > 0$ such that
if $0 < |x - 4| < \delta$ then $|(3x - 5) - 7| < \varepsilon$.

Scratch work: $|(3x - 5) - 7| < \varepsilon$ $\quad |3x - 12| < \varepsilon$ $\quad 3|x - 4| < \varepsilon$ $\quad |x - 4| < \frac{\varepsilon}{3} = \delta$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{3}$. Then
if $0 < |x - 4| < \delta = \frac{\varepsilon}{3}$ then $3|x - 4| < \varepsilon$ or $|3x - 12| < \varepsilon$ or $|(3x - 5) - 7| < \varepsilon$.

10. When you use a left Riemann sum with 3 intervals of equal length to approximate $\ln 7 = \int_1^7 \frac{1}{x}\,dx$ you discover: HINT: Draw a picture.

a. $\ln 7 > \frac{46}{15}$

b. $\ln 7 < \frac{46}{15}$ Correct Choice

c. $\ln 7 > \frac{11}{6}$

d. $\ln 7 < \frac{11}{6}$

e. $\ln 7 < \frac{23}{15}$

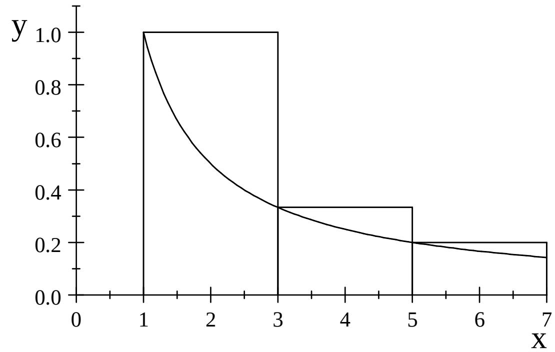
SOLUTION: $\Delta x = \frac{7-1}{3} = 2$

$f(1) = 1 \quad f(3) = \frac{1}{3} \quad f(5) = \frac{1}{5}$

From the plot

$A \approx 1 \cdot 2 + \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 2 = \frac{46}{15}$

The curve is below the rectangles.



11. Evaluate $\int_{-\pi/4}^{\pi/4} \cos(3x)\,dx$

a. $\sqrt{2}$

b. $3\sqrt{2}$

c. 0

d. $\frac{\sqrt{2}}{3}$ Correct Choice

e. $-\sqrt{2}$

SOLUTION:

$$\int_{-\pi/4}^{\pi/4} \cos(3x)\,dx = \left[\frac{\sin(3x)}{3}\right]_{-\pi/4}^{\pi/4} = \frac{\sin\left(\frac{3\pi}{4}\right)}{3} - \frac{\sin\left(-\frac{3\pi}{4}\right)}{3} = \frac{1}{3}\frac{1}{\sqrt{2}} - \frac{1}{3}\frac{-1}{\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

12. If $I = \int_0^2 2^{(x^2)}\,dx$, which of the following is FALSE?

a. $\int_0^2 3 \cdot 2^{(x^2)}\,dx = 3I$

b. $2 \le I \le 32$

c. $\int_0^2 \sqrt{2^{(x^2)}}\,dx = \sqrt{I}$ Correct Choice

d. $\int_0^5 2^{(x^2)}\,dx - \int_2^5 2^{(x^2)}\,dx = I$

e. $\int_2^0 2^{(x^2)}\,dx = -I$

SOLUTION: (a) $\int 3f\,dx = 3\int f\,dx$ - true (d) $\int_0^5 = \int_0^2 + \int_2^5$ - true (e) $\int_2^0 = -\int_0^2$ - true

(b) $1 \le 2^{(x^2)} \le 16$ on $[0,2]$. So $\int_0^2 1\,dx \le \int_0^2 2^{(x^2)}\,dx \le \int_0^2 16\,dx$. Or $2 \le I \le 32$.- true

(c) is false.

13. Compute $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^2} \sin(t^2)\,dt$

- **a**. $\sin(x^4)2x - \sin(x^6)3x^2$ Correct Choice
- **b**. $\cos(x^2)2x - \cos(x^3)3x^2$
- **c**. $\sin(x^4) - \sin(x^6)$
- **d**. $\cos(x^2) - \cos(x^3)$
- **e**. $-\cos(x^4)2x + \cos(x^6)3x$

SOLUTION: Let $F(t)$ be an antiderivative of $\sin(t^2)$.

So $F'(t) = \sin(t^2)$ and $\displaystyle\int_{x^3}^{x^2} \sin(t^2)\,dt = F(x^2) - F(x^3)$.

So $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^2} \sin(t^2)\,dt = \dfrac{d}{dx}[F(x^2) - F(x^3)] = F'(x^2)2x - F'(x^3)3x^2 = \sin(x^4)2x - \sin(x^6)3x^2$

14. Recall: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

Then $\cosh(x)\cosh(2x) - \sinh(x)\sinh(2x) =$

- **a**. $\sinh(3x)$
- **b**. $\sinh(x)$
- **c**. $\cosh(3x)$
- **d**. $\cosh(x)$ Correct Choice
- **e**. $-\cosh(x)$

SOLUTION: $\cosh(x)\cosh(2x) - \sinh(x)\sinh(2x) = \dfrac{e^x + e^{-x}}{2}\dfrac{e^{2x} + e^{-2x}}{2} - \dfrac{e^x - e^{-x}}{2}\dfrac{e^{2x} - e^{-2x}}{2}$

$= \dfrac{e^{3x} + e^x + e^{-x} + e^{-3x}}{4} - \dfrac{e^{3x} - e^x - e^{-x} + e^{-3x}}{4} = \dfrac{2e^x + 2e^{-x}}{4} = \cosh(x)$

15. Find the intervals of concavity of the function $f(x) = 12(12 + x^2)^{-1}$.

- **a**. Concave up: $(2,\infty)$ Concave down: $(-\infty,-2) \cup (-2,2)$
- **b**. Concave up: $(-\infty,-2) \cup (2,\infty)$ Concave down: $(-2,2)$ Correct Choice
- **c**. Concave up: $(-2,2)$ Concave down: $(-\infty,-2) \cup (2,\infty)$
- **d**. Concave up: $(-\infty,\infty)$ Concave down: nowhere
- **e**. Concave up: nowhere Concave down: $(-\infty,\infty)$

SOLUTION: $f'(x) = -12(12 + x^2)^{-2}2x$

$f''(x) = 12 \cdot 2(12 + x^2)^{-3}2x2x - 12 \cdot (12 + x^2)^{-2}2 = 12 \cdot \dfrac{8x^2 - (12 + x^2)2}{(12 + x^2)^3} = 12 \cdot \dfrac{6x^2 - 24}{(12 + x^2)^3} = 0$

$x = \pm 2$

Check the signs in each interval:

$(-\infty,-2)$: $f''(-3) > 0$ concave up

$(-2,2)$: $f''(0) < 0$ concave down

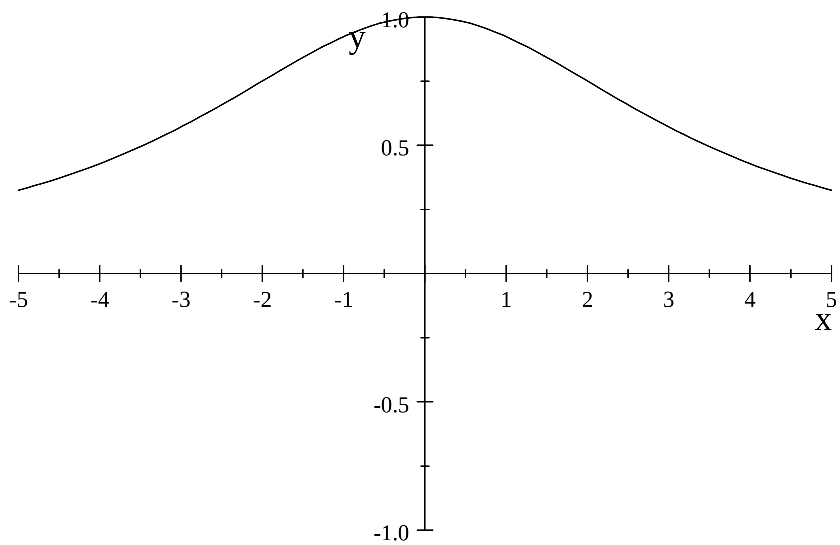
$(2,\infty)$: $f''(3) > 0$ concave up

Part 2 – Work Out Problems (5 questions. Points indicated. No Calculators)

Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (6 points) Graph the function $f(x) = 12(12+x^2)^{-1}$. (See problem #15.)
Be sure to label all maxima, minima, inflection points and asymptotes.
Be careful with intervals of increase, decrease and concavity.

SOLUTION: increasing: $x < 0$ decreasing: $x > 0$
local and absolute maximum: $(0,1)$ no minima
Concave up: $x < -2$ or $x > 2$ Concave down: $-2 < x < 2$
inflection points: $\left(-2, \frac{3}{4}\right)$ and $\left(2, \frac{3}{4}\right)$
no vertical asymptotes, horizontal asymptotes: $y = 0$ as $x \to \pm\infty$



17. (8 points) Let $g(x)$ be the inverse function of $f(x) = \frac{1}{\ln x}$ for $x > 0$. Find $g(1)$ and $g'(1)$.

SOLUTION: If $b = g(1)$ then $1 = f(b) = \frac{1}{\ln b}$. So $b = e$. So $g(1) = e$.

Further $f'(x) = \frac{-1}{(\ln x)^2}\frac{1}{x}$ and $f'(e) = \frac{-1}{(\ln e)^2}\frac{1}{e} = \frac{-1}{e}$. So $g'(1) = \frac{1}{f'(e)} = -e$.

18. (10 points) The pressure and volume of the gas in an engine cylinder are related by $PV^{3/2} = C$ for some constant C. Currently, the pressure is $P = 15$ lb/in^2 and the volume is $V = 4$ in^3 but is increasing at the rate of $\frac{dV}{dt} = 2$ in^3/min. Find C and the current value of $\frac{dP}{dt}$. Is the pressure currently increasing or decreasing?

SOLUTION: $C = PV^{3/2} = 15 \cdot 4^{3/2} = 15 \cdot 8 = 120$

$P\frac{3}{2}V^{1/2}\frac{dV}{dt} + \frac{dP}{dt}V^{3/2} = 0 \qquad \frac{dP}{dt}V^{3/2} = -P\frac{3}{2}V^{1/2}\frac{dV}{dt} \qquad \frac{dP}{dt} = -\frac{P}{V}\frac{3}{2}\frac{dV}{dt} = -\frac{15}{4}\frac{3}{2}2 = -\frac{45}{4}$

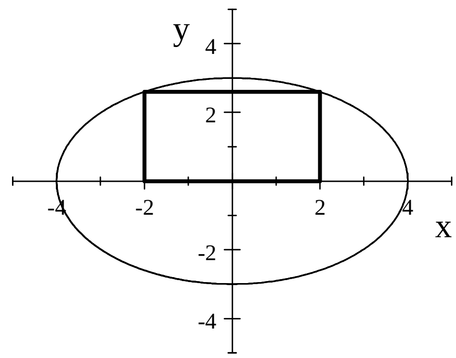
P is decreasing.

19. (6 points) Find a parametric equation for the line perpendicular to the parametric curve $\vec{r}(t) = \langle t^2, t^3 \rangle$ at the point $(4,8)$.

SOLUTION: $t = 2 \qquad \vec{v}(t) = \langle 2t, 3t^2 \rangle \qquad \vec{v}(2) = \langle 4, 12 \rangle \qquad \vec{n} = \langle 12, -4 \rangle$

$\vec{r}_{\tan}(t) = (4,8) + t\langle 12,-4 \rangle = \langle 4 + 12t, 8 - 4t \rangle$ There is no unique answer.

20. (12 points) A rectangle is inscribed in the upper half of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with its base on the x-axis. Find the maximum area of such a rectangle.



SOLUTION: $A = 2xy$ with $y = 3\sqrt{1 - \frac{x^2}{16}}$. So $A = 6x\sqrt{1 - \frac{x^2}{16}}$

$A' = 6\sqrt{1 - \frac{x^2}{16}} + 6x\frac{1}{2}\frac{\frac{-2x}{16}}{\sqrt{1 - \frac{x^2}{16}}} = 0$ Multiply by $\frac{1}{6}\sqrt{1 - \frac{x^2}{16}}$

$\left(1 - \frac{x^2}{16}\right) - \frac{x^2}{16} = 0$ or $1 - \frac{x^2}{8} = 0 \qquad x = \sqrt{8} = 2\sqrt{2}$

$y = 3\sqrt{1 - \frac{8}{16}} = \frac{3}{\sqrt{2}} \qquad A = 2xy = 2\left(2\sqrt{2}\right)\left(\frac{3}{\sqrt{2}}\right) = 12$