Student (Print) ______________________________ Section __________

Last, First Middle

Student (Sign) ______________________________

Student ID ______________________________

Instructor ______________________________

MATH 152

Exam 2

Fall 2000

Test Form B

Solutions

1-10	/50
11	/10
12	/10
13	/10
14	/10
15	/10
TOTAL	

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

Formulas:

$$M_n = \Delta x\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

$$T_n = \frac{\Delta x}{2}[f(x_o)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]$$

$$S_n = \frac{\Delta x}{3}[f(x_o)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

$|E_M| < \dfrac{K(b-a)^3}{24n^2}$ where $K > f''(x)$ for all x in $[a,b]$

$|E_T| < \dfrac{K(b-a)^3}{12n^2}$ where $K > f''(x)$ for all x in $[a,b]$

$|E_S| < \dfrac{K(b-a)^5}{180n^4}$ where $K > f^{(4)}(x)$ for all x in $[a,b]$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute $\int_e^\infty \frac{1}{x\ln x}\,dx$.

- **a.** 0
- **b.** $\frac{1}{e}$
- **c.** 1
- **d.** e
- **e.** ∞ correctchoice

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$\int_e^\infty \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x|\Big|_e^\infty = \lim_{b\to\infty} \ln|\ln b| - \ln|\ln e| = \infty$

2. After separating variables, we can solve the differential equation $\frac{dy}{dt} = ty + \frac{t}{y}$ by solving

- **a.** $\int\left(\frac{1}{y} + y\right)dy = \int t\,dt$
- **b.** $\int \frac{y}{y^2+1}\,dy = \int t\,dt$ correctchoice
- **c.** $\int (y^2 + y)\,dy = \int (t+1)\,dt$
- **d.** $\int \frac{y^2+1}{y}\,\frac{1}{dy} = \int \frac{t}{dt}$
- **e.** None of the above, the differential equation is not separable.

$\frac{dy}{dt} = t\left(y + \frac{1}{y}\right) = t\left(\frac{y^2+1}{y}\right) \quad \Rightarrow \quad \frac{y}{y^2+1}\,dy = t\,dt \quad \Rightarrow \quad \int \frac{y}{y^2+1}\,dy = \int t\,dt$

3. Compute $\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n}$

- **a.** 0
- **b.** $\frac{1}{3}$ correctchoice
- **c.** $\frac{2}{3}$
- **d.** $\frac{\ln 2}{3}$
- **e.** ∞

Use l'Hopital's Rule twice: $\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} = \lim_{n\to\infty} \frac{\frac{e^n}{2+e^n}}{3} = \frac{1}{3}\lim_{n\to\infty} \frac{e^n}{2+e^n} = \frac{1}{3}\lim_{n\to\infty} \frac{e^n}{e^n} = \frac{1}{3}$

OR: Since e^n is much greater than 2, $\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} = \lim_{n\to\infty} \frac{\ln(e^n)}{3n} = \lim_{n\to\infty} \frac{n}{3n} = \frac{1}{3}$

4. Which of the following gives the Trapezoid Rule approximation to $\int_2^4 \frac{1}{x}\,dx$ with $n = 4$?

a. $\frac{1}{4}\left[\frac{1}{4} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{8}\right]$
b. $\frac{1}{4}\left[\frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right]$
c. $\frac{1}{4}\left[\frac{1}{2} + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{4}\right]$ correctchoice
d. $\frac{1}{2}\left[\ln 2 + \ln\frac{3}{2} + \ln 3 + \ln\frac{7}{2} + \ln 4\right]$
e. $\frac{1}{4}\left[\ln 2 + 2\ln\frac{3}{2} + 2\ln 3 + 2\ln\frac{7}{2} + \ln 4\right]$

$\Delta x = \frac{4-2}{4} = \frac{1}{2}$
$T4 = \frac{\Delta x}{2}\left[f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right]$
$T4 = \frac{1}{4}\left[\frac{1}{2} + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{4}\right]$

5. If you use the Trapezoid Rule with $n = 4$ to approximate $\int_2^4 \frac{1}{x}\,dx$, which of the following is the smallest upper bound on the error $|E_T|$ in the approximation?

a. $\frac{1}{12}$
b. $\frac{1}{24}$
c. $\frac{1}{48}$
d. $\frac{1}{96}$ correctchoice
e. $\frac{1}{192}$

$f''(x) = \frac{2}{x^3}$ On $[2,4]$ the maximum is $K = f''(2) = \frac{2}{8} = \frac{1}{4}$. So the error is bounded by

$|E_T| < \frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{4}(4-2)^3}{12(4)^2} = \frac{1}{96}$

6. An integrating factor for the differential equation $\frac{dy}{dx} = x\sin 2x + y\tan x$ is $I(x) =$

a. $|\cos x|$ correctchoice
b. $e^{-\sec^2 x}$
c. $e^{-|\tan x|}$
d. $-|\sec x|$
e. $e^{|\cos x|}$

Standard form: $\frac{dy}{dx} - (\tan x)y = x\sin 2x$ $P = -\tan x$
$\int P\,dx = -\int \tan x\,dx = -\int \frac{\sin x}{\cos x}\,dx = \ln|\cos x|$
$I(x) = e^{\int P\,dx} = e^{\ln|\cos x|} = |\cos x|$

7. Find the arc length of the parametric curve $x = 4\cos t, \quad y = 4\sin t$ for $\frac{\pi}{6} \le t \le \frac{\pi}{3}$.

a. $\frac{\pi}{3}$
b. $\frac{2\pi}{3}$ correctchoice
c. $\frac{\pi}{2}$
d. $\frac{4\pi}{3}$
e. $\frac{3\pi}{4}$

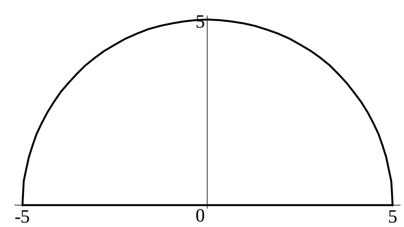
$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 4\cos t$$

$$L = \int_{\pi/6}^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{\pi/6}^{\pi/3} \sqrt{16\sin^2 t + 16\cos^2 t}\, dt = \int_{\pi/6}^{\pi/3} 4\, dt = 4\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2\pi}{3}$$

OR: $\frac{\pi/3 - \pi/6}{2\pi} 2\pi 4 = \frac{2\pi}{3}$

8. A tank is completely filled with water. The end is a vertical semicircle with radius 5 m as shown. Which integral gives the hydrostatic force on this end of the tank?

The density of water is $1000\ \frac{\text{kg}}{\text{m}^3}$ and $g = 9.8\frac{\text{m}}{\text{sec}^2}$.



a. $9800\int_0^5 (5-y)2\sqrt{25-y^2}\, dy$ correctchoice
b. $9800\int_0^5 (5+y)\sqrt{25-y^2}\, dy$
c. $9800\int_0^5 (5-y)(25-y^2)\, dy$
d. $9800\int_0^5 (5+y)2\sqrt{25-y^2}\, dy$
e. $9800\int_0^5 (5+y)(25-y^2)\, dy$

The slice at height y is at a distance $h = 5 - y$ below the surface. Its width is $w = 2x$ where $x = \sqrt{25-y^2}$

$$F = \int \rho g h w\, dy = 9800\int_0^5 (5-y)2\sqrt{25-y^2}\, dy$$

9. The parametric curve $x = e^t - t, \quad y = 4e^{t/2}$ for $0 \le t \le 2$ is rotated about the x-axis. Which integral gives the area of the surface of revolution?
HINT: Look for a perfect square.

a. $\int_0^2 2\pi(e^t - t)(e^{2t} + 2e^t + 1)\,dt$

b. $\int_0^2 2\pi(e^t - t)(e^t + 1)\,dt$

c. $\int_0^2 8\pi e^{t/2}(e^{2t} + 2e^t + 1)\,dt$

d. $\int_0^2 8\pi e^{t/2}(e^t + 1)\,dt$ correctchoice

e. $\int_0^2 8\pi e^{t/2}\sqrt{e^t + 1}\,dt$

$$\frac{dx}{dt} = e^t - 1 \qquad \frac{dy}{dt} = 2e^{t/2}$$

$$A = \int 2\pi r\,ds = \int_0^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 2\pi 4e^{t/2}\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt$$

$$= \int_0^2 8\pi e^{t/2}\sqrt{(e^{2t} - 2e^t + 1) + (4e^t)}\,dt = \int_0^2 8\pi e^{t/2}\sqrt{e^{2t} + 2e^t + 1}\,dt = \int_0^2 8\pi e^{t/2}(e^t + 1)\,dt$$

10. Calculate the x-component (or coordinate) of the center of mass of a plate with uniform density ρ and whose shape is the quarter circle of radius 3 given by $0 \le x \le 3$ and $0 \le y \le \sqrt{9 - x^2}$.

a. $\frac{\pi}{4}$

b. $\frac{9\pi}{4}\rho$

c. 9ρ

d. $\frac{8}{\pi}$

e. $\frac{4}{\pi}$ correctchoice

$$M = \int_0^3 \rho\sqrt{9 - x^2}\,dx = \rho\frac{1}{4}\pi r^2 = \frac{9\pi}{4}\rho$$

$$M_y = \int_0^3 \rho x\sqrt{9 - x^2}\,dx = -\rho\frac{1}{3}(9 - x^2)^{3/2}\Big|_0^3 = \rho\frac{1}{3}9^{3/2} = 9\rho$$

$$\bar{x} = \frac{M_y}{M} = \frac{9\rho \bullet 4}{9\pi\rho} = \frac{4}{\pi}$$

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.

You may not use a calculator.

11. Determine whether each of the following integrals converges or diverges. If it converges, find its value.
Hint: Partial fractions

a. $\int_1^\infty \frac{2}{2x + x^2}\,dx$ Circle one: [Converges] Diverges

Explain:

Partial fractions gives $\frac{2}{2x + x^2} = \frac{1}{x} - \frac{1}{2+x}$

$$\int_1^\infty \frac{2}{2x+x^2}\,dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{2+x}\right) dx = \Big[\ln x - \ln(2+x)\Big]_1^\infty = \left[\ln\frac{x}{2+x}\right]_1^\infty = \ln 1 - \ln\frac{1}{3} = \ln 3$$

b. $\int_0^1 \frac{2}{2x + x^2}\,dx$ Circle one: Converges [Diverges]

Explain:

$$\int_0^1 \frac{2}{2x+x^2}\,dx = \int_0^1 \left(\frac{1}{x} - \frac{1}{2+x}\right) dx = \Big[\ln x - \ln(2+x)\Big]_0^1 = [0 - \ln 2] - [-\infty - 0] = \infty$$

12. Determine whether each of the following sequences converges or diverges. If it converges, find the limit. Fully justify your answers.

a. $\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ Circle one: Converges [Diverges]

Explain:

$\cos\left(\frac{n\pi}{2}\right)$ takes the values $0, -1, 0, 1$ repeatedly. So it can never have a limit.

b. $\lim_{n\to\infty} \frac{\sin^2 n}{2^n}$ Circle one: [Converges] Diverges

Explain:

$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$ and $\lim_{n\to\infty} \frac{1}{2^n} = 0$. By the sandwich theorem, $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$.

13. Solve the initial value problem

$$x\frac{dy}{dx} + 2y = \frac{\cos x}{x} \qquad \text{with} \qquad y(\pi) = 2.$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2} \qquad P = \frac{2}{x} \qquad I = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = \cos x \qquad \frac{d}{dx}(x^2 y) = \cos x \qquad x^2 y = \int \cos x\, dx = \sin x + C$$

$x = \pi$ when $y = 2$ $\qquad 2\pi^2 = \sin\pi + C \qquad C = 2\pi^2 \qquad x^2 y = \sin x + 2\pi^2 \qquad y = \dfrac{\sin x + 2\pi^2}{x^2}$

14. Compute the surface area of the surface obtained by rotating the curve $x = 1 + y^2$ for $1 \le y \le 2$ about the x-axis?

$$A = \int 2\pi r\, ds = \int_1^2 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 2\pi y\sqrt{1 + 4y^2}\, dy$$

$$u = 1 + 4y^2 \qquad du = 8y\, dy \qquad \frac{du}{8} = y\, dy$$

$$A = \frac{1}{8}\int_{y=1}^{2} 2\pi\sqrt{u}\, du = \left[\frac{\pi}{4}\frac{2u^{3/2}}{3}\right]_{y=1}^{2} = \left[\frac{\pi}{6}(1 + 4y^2)^{3/2}\right]_1^2$$

$$= \frac{\pi}{6}(17)^{3/2} - \frac{\pi}{6}(5)^{3/2}$$

15. A tank contains 40 lb of salt mixed with 60 gal of water. Salt water containing 4 lb of salt per gal is added to the tank at the rate of 3 gal per min. The tank is kept thoroughly mixed and drains at the same rate.

a. Write out the differential equation and the initial condition for $S(t)$, the number of lbs of salt in the tank at time t.

$$\frac{dS}{dt}\frac{\text{lb}}{\text{min}} = \frac{4\text{ lb}}{\text{gal}}\frac{3\text{ gal}}{\text{min}} - \frac{S(t)\text{ lb}}{60\text{ gal}}\frac{3\text{ gal}}{\text{min}} \qquad \frac{dS}{dt} = 12 - \frac{1}{20}S \qquad S(0) = 40$$

b. Solve the initial value problem.

$$\frac{dS}{dt} = 12 - \frac{1}{20}S \qquad \int\frac{dS}{12 - \frac{1}{20}S} = \int dt \qquad -20\ln\left|12 - \frac{1}{20}S\right| = t + C$$

$$\ln\left|12 - \frac{1}{20}S\right| = -\frac{t}{20} - \frac{C}{20} \qquad 12 - \frac{1}{20}S = Ae^{-t/20} \qquad S = 240 - 20Ae^{-t/20}$$

$$S(0) = 40 = 240 - 20A \qquad A = 10 \qquad S = 240 - 200e^{-t/20}$$

c. How much salt is in the tank after 20 hours?

$$S(1200) = 240 - 200e^{-1200/20} = 240 - 200e^{-60}$$

The problem should have said 20 min which gives

$$S(20) = 240 - 200e^{-20/20} = 240 - \frac{200}{e}$$