Student (Print) ______________________________ Section __________

Last, First Middle

Student (Sign) ______________________________

Student ID ______________________________

Instructor ______________________________

MATH 152
Exam 3
Fall 2000
Test Form A
Solutions

1-9	/45
10	/10
11	/15
12	/10
13	/10
14	/10
TOTAL	

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute $\sum_{n=0}^{\infty} 3^n 5^{2-n}$

a. $\frac{25}{3}$
b. $\frac{125}{3}$
c. $\frac{125}{2}$ correctchoice
d. $\frac{75}{2}$
e. ∞

Geometric: $a = 3^0 5^2 = 25$, $r = \frac{3}{5} < 1$ $S = \frac{25}{1 - \frac{3}{5}} = \frac{125}{2}$

2. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is

a. absolutely convergent
b. convergent but not absolutely convergent correctchoice
c. divergent
d. none of the above

The series is convergent by the Alternating Series Test since it is (1) alternating due to the $(-1)^n$, (2) decreasing in absolute value since $\frac{1}{n}$ gets smaller, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

However it is not absolutely convergent because the related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series which is divergent.

3. Determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{n+1}$.

a. $(2,4]$
b. $[2,4)$
c. $(-4,-2]$
d. $[-4,-2)$ correctchoice
e. $(-\infty,\infty)$

Ratio Test: $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x+3|^{n+1}}{n+2} \frac{n+1}{|x+3|^n} = |x+3| < 1$ or $-4 < x < -2$

At $x = -4$ the series is alternating and decreasing and so convergent.
At $x = -2$ the series is harmonic and so divergent.
So the interval of convergence is $-4 \le x < -2$.

4. Find the Maclaurin series for $f(x) = \sin 2x$.

a. $2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \cdots$ correctchoice

b. $2x - \frac{2x^3}{3!} + \frac{2x^5}{5!} - \frac{2x^7}{7!} + \cdots$

c. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

d. $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

e. $2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$

The Maclaurin series for $\sin x$ is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Substitute $x \to 2x$: $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \cdots$

5. Find the 3rd degree Taylor polynomial for $f(x) = e^{-2x}$ about $x = 1$.

a. $e^{-2} + 2e^{-2}(x-1) + 2e^{-2}(x-1)^2 + \frac{4}{3}e^{-2}(x-1)^3$

b. $e^{-2} - 2e^{-2}(x-1) + 2e^{-2}(x-1)^2 - \frac{4}{3}e^{-2}(x-1)^3$ correctchoice

c. $e^{-2} + 2e^{-2}(x-1) + 4e^{-2}(x-1)^2 + 8e^{-2}(x-1)^3$

d. $e^{-2} - 2e^{-2}(x-1) + 4e^{-2}(x-1)^2 - 8e^{-2}(x-1)^3$

e. $e^{-2} - 2e^{-2}(x-1) - 4e^{-2}(x-1)^2 - 8e^{-2}(x-1)^3$

The function, 3 derivatives and their values at $x = 1$ are

$$\begin{aligned} f(x) &= e^{-2x} & f(1) &= e^{-2} \\ f'(x) &= -2e^{-2x} & f'(1) &= -2e^{-2} \\ f''(x) &= 4e^{-2x} & f''(1) &= 4e^{-2} \\ f'''(x) &= -8e^{-2x} & f'''(1) &= -8e^{-2} \end{aligned}$$

So the Taylor polynomial is

$$\begin{aligned} T_3 &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \\ &= e^{-2} - 2e^{-2}(x-1) + \frac{4e^{-2}}{2}(x-1)^2 - \frac{8e^{-2}}{6}(x-1)^3 \end{aligned}$$

6. Compute $\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$. HINT: Differentiate $\sum_{n=0}^{\infty} x^n$.

a. -4

b. -2

c. $\frac{4}{9}$

d. 2

e. 4 correctchoice

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ provided $|x| < 1$

Differentiate: $\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ provided $|x| < 1$

Evaluate at $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$

7. Find a unit vector which is orthogonal to the plane containing the points $P = (1,0,0)$, $Q = (0,2,0)$, and $R = (0,0,3)$.

- **a.** $\left(\frac{6}{11}, \frac{-3}{11}, \frac{2}{11}\right)$
- **b.** $(6,-3,2)$
- **c.** $(6,3,2)$
- **d.** $\left(\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}\right)$
- **e.** $\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$ correctchoice

$\overrightarrow{PQ} = Q - P = (-1,2,0)$, $\overrightarrow{PR} = R - P = (-1,0,3)$

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6,3,2) \qquad \left|\vec{N}\right| = \sqrt{36+9+4} = \sqrt{49} = 7 \qquad \frac{\vec{N}}{\left|\vec{N}\right|} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

8. Find the equation of the circle for which one diameter has endpoints $P = (1,1)$ and $Q = (7,9)$.

- **a.** $(x+3)^2 + (y+4)^2 = 25$
- **b.** $(x+3)^2 + (y+4)^2 = 5$
- **c.** $(x-4)^2 + (y-5)^2 = 25$ correctchoice
- **d.** $(x-3)^2 + (y-4)^2 = 5$
- **e.** $(x+4)^2 + (y+5)^2 = 25$

The center is at $C = \frac{P+Q}{2} = (4,5)$ and the radius is $r = \left|\overrightarrow{CP}\right| = |(3,4)| = 5$.

9. Consider the triangle with vertices $A = (0,0,0)$, $B = (1,1,0)$ and $C = (0,2,2)$. Find the angle at A.

- **a.** $0°$
- **b.** $30°$
- **c.** $45°$
- **d.** $60°$ correctchoice
- **e.** $90°$

$$\left|\overrightarrow{AB}\right| = \sqrt{1+1} = \sqrt{2} \qquad \left|\overrightarrow{AC}\right| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \qquad \overrightarrow{AB} \bullet \overrightarrow{AC} = 2$$

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{2}{\sqrt{2}\,2\sqrt{2}} = \frac{1}{2} \qquad \theta = 60°$$

Part II: Work Out

Show all your work. Partial credit will be given.

You may not use a calculator.

10. (10 points)

a. Find the power series representation for $f(x) = \dfrac{1}{1+x^2}$ centered at $x = 0$ and its radius of convergence.

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ provided $|x| < 1$

Substitute $x \to -x^2$

$\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ provided $|x| < 1$ So $R = 1$

b. Find the power series representation for $f(x) = x\tan^{-1}x$ centered at $x = 0$ and its radius of convergence. HINT: $\tan^{-1}x = \int_0^x \dfrac{1}{1+t^2}\,dt$

$\tan^{-1}x = \int_0^x \dfrac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ and $R = 1$

$f(x) = x\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{2n+1}$ and $R = 1$

11. (15 points) Determine if each of the following series converges or diverges. Say why. Be sure to name or quote the test(s) you use and check out all requirements of the test.

a. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ Circle one: **[Converges]** Diverges

Explain:

The related absolute series is $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Apply the Integral Test: $\frac{1}{n(\ln n)^2}$ is continuous, decreasing and

$$\int_2^{\infty} \frac{1}{n(\ln n)^2} = \left[\frac{-1}{\ln n} \right]_2^{\infty} = 0 - \frac{-1}{\ln 2} = \frac{1}{\ln 2}$$

So $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ also converges.

b. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ Circle one: **[Converges]** Diverges

Explain:

Apply Alternating Series Test:

It is alternating because of the $(-3)^n$. It is decreasing in absolute value because $\frac{3^n}{n!}$ gets smaller as n gets larger provided $n > 3$. Finally, $\lim_{n\to\infty} \frac{3^n}{n!} = \lim_{n\to\infty} \frac{3 \bullet 3 \bullet 3 \bullet \cdots \bullet 3}{1 \bullet 2 \bullet 3 \bullet \cdots \bullet n} = 0$ because beyond $n = 3$ we keep multiplying by fractions less than 1. So $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ converges.

OR

Apply the Ratio Test:

$|a_n| = \frac{3^n}{n!}$ $\quad |a_{n+1}| = \frac{3^{n+1}}{(n+1)!}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

So $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ converges.

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{n-1}}{n^2 4^n}$ Circle one: Converges **[Diverges]**

Explain:

Apply the Ratio Test:

$|a_n| = \frac{5^{n-1}}{n^2 4^n}$ $\quad |a_{n+1}| = \frac{5^n}{(n+1)^2 4^{n+1}}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{5^n}{(n+1)^2 4^{n+1}} \frac{n^2 4^n}{5^{n-1}} = \frac{5}{4} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{5}{4} > 1$$

So $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ diverges.

12. (10 points) Determine if each of the following series converges or diverges. Say why.
Be sure to name or quote the test(s) you use and check out all requirements of the test.
If it converges, find the sum. **If it diverges**, does it diverge to $+\infty$, $-\infty$ or neither?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt[3]{n}}$ Circle one: Converges **[Diverges]**

Explain:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges because it is a p-series with $p = \frac{1}{3} < 1$.

Apply the Limit Comparison Test: $a_n = \dfrac{1}{1+\sqrt[3]{n}}$ $b_n = \dfrac{1}{\sqrt[3]{n}}$

$L = \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{1+\sqrt[3]{n}} = 1$ and $0 < L < \infty$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt[3]{n}}$ also diverges. Since $\dfrac{1}{1+\sqrt[3]{n}} > 0$, it diverges to $+\infty$.

b. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right)$ Circle one: **[Converges]** Diverges

Explain:

$$S_k = \sum_{n=1}^{k} \left(\frac{n-1}{n} - \frac{n}{n+1}\right) = \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{2}{3}\right) + \cdots + \left(\frac{k-1}{k} - \frac{k}{k+1}\right) = \frac{-k}{k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{-k}{k+1} = -1$$

13. (10 points) Find the volume of the parallelepiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$, where

$$P = (1,1,1), \quad Q = (2,3,1), \quad R = (4,1,5) \text{ and } S = (1,3,4).$$

$\overrightarrow{PQ} = Q - P = (1,2,0)$, $\overrightarrow{PR} = R - P = (3,0,4)$ and $\overrightarrow{PS} = S - P = (0,2,3)$

$$V = \overrightarrow{PQ} \bullet \left(\overrightarrow{PR} \times \overrightarrow{PS}\right) = \left|\det\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}\right| = |-8 - 18| = 26$$

14. (10 points) Determine the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{1/3}3^n}$. Be sure to show how you checked the convergence at the endpoints.

Apply the Ratio Test:

$$|a_n| = \frac{|x-2|^n}{n^{1/3}3^n} \qquad |a_{n+1}| = \frac{|x-2|^{n+1}}{(n+1)^{1/3}3^{n+1}}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-2|^{n+1}}{(n+1)^{1/3}3^{n+1}}\frac{n^{1/3}3^n}{|x-2|^n} = \frac{|x-2|}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/3} = \frac{|x-2|}{3}$$

The series converges when $L = \frac{|x-2|}{3} < 1$ or $|x-2| < 3$ or $-1 < x < 5$

We check the endpoint at $x = -1$.

The series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^{1/3}3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$ which is convergent because it is an alternating, decreasing series.

We check the endpoint at $x = 5$.

The series is $\sum_{n=1}^{\infty}\frac{(3)^n}{n^{1/3}3^n} = \sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ which is a divergent p-series since $p = \frac{1}{3} < 1$.

Thus the interval of convergence is $-1 \le x < 5$ or $[-1,5)$.