1-12	/48
13	/13
14	/13
15	/13
16	/13

MATH 152 Final Exam Fall 2000

Sections 510,511 Solutions P. Yasskin

Multiple Choice: (4 points each)

1. Compute $\lim_{n\to\infty} \frac{2^{n+1}}{n!}$

a. 0 correctchoice

b. 1

c. 2

d. 4

e. ∞

$$\lim_{n\to\infty} \frac{2^{n+1}}{n!} = \lim_{n\to\infty} \frac{\overbrace{2 \bullet 2 \bullet 2 \bullet 2 \bullet \cdots \bullet 2}^{n+1}}{1 \bullet 2 \bullet 3 \bullet \cdots \bullet n} = 2 \lim_{n\to\infty} \frac{2}{1} \bullet \frac{2}{2} \bullet \frac{2}{3} \bullet \cdots \bullet \frac{2}{n} < 2 \lim_{n\to\infty} \frac{2}{1} \bullet \frac{2}{2} \bullet \left(\frac{2}{3}\right)^{n-2} = 0$$

since all but the first 2 factors are less than $\frac{2}{3}$.

2. Compute $\int_0^{\pi/4} \tan\theta \sec^2\theta \, d\theta$

a. $-\frac{1}{2}$

b. $-\frac{1}{3}$

c. 0

d. $\frac{1}{3}$

e. $\frac{1}{2}$ correctchoice

$u = \tan\theta \qquad du = \sec^2\theta \, d\theta \qquad \int_0^{\pi/4} \tan\theta \sec^2\theta \, d\theta = \int_0^1 u \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

3. The region below $y = \frac{1}{x}$ above the x-axis between $x = 1$ and $x = 2$ is rotated about the y-axis. Find the volume of the solid of revolution.

a. 4π

b. 2π correctchoice

c. π

d. $\frac{\pi}{2}$

e. $\frac{\pi}{4}$

Use x-integral and cylinders. $V = \int 2\pi r h \, dx = \int_1^2 2\pi x \frac{1}{x} dx = 2\pi$

4. Compute $\int_0^{1/2} x\cos(\pi x)\,dx$

a. $\frac{1}{2\pi} - \frac{1}{\pi^2}$ correctchoice
b. $\frac{1}{\pi^2} + \frac{1}{2\pi}$
c. $\pi^2 - 2\pi$
d. $-2\pi - \pi^2$
e. $\frac{1}{\pi^2} - \frac{1}{2\pi}$

$$\int_0^{1/2} x\cos(\pi x)\,dx = x\frac{\sin(\pi x)}{\pi}\Big|_0^{1/2} - \int_0^{1/2} \frac{\sin(\pi x)}{\pi}\,dx \qquad u = x \quad dv = \cos(\pi x)\,dx \quad du = dx \quad v = \frac{\sin(\pi x)}{\pi}$$

$$= \left[x\frac{\sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi^2}\right]_0^{1/2} = \left[\frac{1}{2}\frac{\sin\left(\frac{\pi}{2}\right)}{\pi} + \frac{\cos\left(\frac{\pi}{2}\right)}{\pi^2}\right] - \left[0 + \frac{\cos(0)}{\pi^2}\right] = \frac{1}{2\pi} - \frac{1}{\pi^2}$$

5. Find the volume of the parallelepiped whose adjacent edges are

$$\vec{u} = (2,-1,3), \quad \vec{v} = (0,1,2) \quad \text{and} \quad \vec{w} = (3,0,1).$$

a. 1
b. -1
c. 13 correctchoice
d. -13
e. $10\sqrt{7}$

$$V = |\vec{u}\cdot\vec{v}\times\vec{w}| = \left|\det\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}\right| = |-13| = 13$$

6. Compute $\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2}{x^4}$

a. $-\frac{1}{6}$
b. $-\frac{1}{2}$
c. 0
d. $\frac{1}{2}$ correctchoice
e. $\frac{1}{6}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \qquad e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots \qquad e^{x^2} - 1 - x^2 = \frac{x^4}{2} + \frac{x^6}{6} + \cdots$$

$$\lim_{x\to 0}\frac{e^{x^2} - 1 - x^2}{x^4} = \lim_{x\to 0}\frac{\frac{x^4}{2} + \frac{x^6}{6} + \cdots}{x^4} = \lim_{x\to 0}\left(\frac{1}{2} + \frac{x^2}{6} + \cdots\right) = \frac{1}{2}$$

7. A 50 gal tank is initially filled with salt water whose concentration is $0.6\frac{\text{lb sugar}}{\text{gal water}}$. Pure water is added at the rate of $5\frac{\text{gal}}{\text{hr}}$. The mixture is kept well mixed and drained at the rate of $5\frac{\text{gal}}{\text{hr}}$. How many hours does it take until the concentration drops to $0.3\frac{\text{lb sugar}}{\text{gal water}}$.

a. $T = -10\ln\frac{1}{2}$ correctchoice

b. $T = 10\ln\frac{1}{2}$

c. $T = \ln 5$

d. $T = -\frac{1}{10}\ln 2$

e. $T = \frac{1}{10}\ln 2$

$$\frac{dS}{dt}\frac{\text{lb}}{\text{hr}} = -5\frac{\text{gal}}{\text{hr}}\frac{S\text{ lb}}{50\text{ gal}} = -\frac{1}{10}S \quad\Rightarrow\quad S = Ae^{-t/10}$$

$$S(0) = 0.6\frac{\text{lb}}{\text{gal}}50\text{ gal} = 30\text{ lb} \quad\Rightarrow\quad S = 30e^{-t/10}$$

$$S(T) = 0.3\frac{\text{lb}}{\text{gal}}50\text{ gal} = 15\text{ lb} \quad\Rightarrow\quad 15 = 30e^{-T/10}$$

$$\Rightarrow\quad e^{-T/10} = \frac{1}{2} \quad\Rightarrow\quad T = -10\ln\frac{1}{2}$$

8. Compute $\int_2^3 \frac{1}{(x-2)^{4/3}}\,dx$

a. $-\infty$

b. -3

c. -1

d. 3

e. ∞ correctchoice

$$\int_2^3 \frac{1}{(x-2)^{4/3}}\,dx = \int_2^3 (x-2)^{-4/3}\,dx = -3(x-2)^{-1/3}\Big|_2^3 = -3 + 3\lim_{x\to 2^+}\frac{1}{(x-2)^{1/3}} = +\infty$$

9. The curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. Which formula will give the surface area of the surface of revolution?

a. $A = \int_0^\pi 2\pi x\sqrt{1+\cos^2 x}\,dx$

b. $A = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$

c. $A = \int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}\,dx$ correctchoice

d. $A = \int_0^\pi 2\pi x\sqrt{1+\sin^2 x}\,dx$

e. $A = \int_0^\pi \sqrt{1+\sin^2 x}\,dx$

$$A = \int_0^\pi 2\pi r\,ds = \int_0^\pi 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\,dx$$

10. Find the solution of the differential equation $x\dfrac{dy}{dx} - 2y = 4x^3$ satisfying the initial condition $y(2) = 4$.

a. $y = 4x^3 - \dfrac{127}{8}x^2$
b. $y = 4x^3 - 28$
c. $y = 4x^3 - 7x^2$ correctchoice
d. $y = \dfrac{4}{5}x^3 - \dfrac{12}{5}$
e. $y = \dfrac{4}{5}x^3 - \dfrac{48}{5x^2}$

Standard form: $\dfrac{dy}{dx} - \dfrac{2}{x}y = 4x^2$

$$P = -\frac{2}{x} \quad \Rightarrow \quad \int P\,dx = -\int \frac{2}{x}\,dx = -2\ln x \quad \Rightarrow \quad I = e^{\int P\,dx} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = 4 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{y}{x^2}\right) = 4 \quad \Rightarrow \quad \frac{y}{x^2} = 4x + C$$

Initial Condition: $x = 2,\ \ y = 4 \quad \Rightarrow \quad \dfrac{4}{2^2} = 4(2) + C \quad \Rightarrow \quad C = -7$

$$\Rightarrow \quad \frac{y}{x^2} = 4x - 7 \quad \Rightarrow \quad y = 4x^3 - 7x^2$$

11. Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{3^n}(x-2)^n$.

a. 0
b. $\dfrac{1}{2}$
c. 2
d. $\dfrac{1}{3}$
e. 3 correctchoice

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)^2(x-2)^{n+1}}{3^{n+1}}\,\frac{3^n}{(n+1)^2(x-2)^n}\right| = \frac{|x-2|}{3}\lim_{n\to\infty}\left|\frac{(n+2)^2}{(n+1)^2}\right|$$

$$L = \frac{|x-2|}{3} < 1 \quad \Rightarrow \quad |x-2| < 3 \quad \Rightarrow \quad R = 3$$

12. Find an equation of the plane perpendicular to the line

$$x = 2 - 2t \quad y = -1 + 3t \quad z = 3 + t$$

which contains the point $P = (1,2,3)$

a. $2x - y + 3z = 9$
b. $-2x + 3y + z = 7$ correctchoice
c. $x + 2y + 3z = 14$
d. $2x + y + z = 7$
e. $2x + 2y + z = 9$

The normal to the plane is the tangent to the line: $\vec{N} = \vec{v} = (-2,3,1)$.
So the equation of the plane is

$$\vec{N} \bullet (X - P) = 0 \quad \Rightarrow \quad (-2,3,1) \bullet (x-1, y-2, z-3) = 0 \quad \Rightarrow \quad -2x + 3y + z - 7 = 0$$

Work Out (13 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

13. Compute $\int \frac{32}{(x-2)(x+2)(x^2+4)}\,dx$

$$\frac{32}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$32 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)$$

$x = 2$: $\quad 32 = A(4)(8) \quad \Rightarrow \quad A = 1$

$x = -2$: $\quad 32 = B(-4)(8) \quad \Rightarrow \quad B = -1$

$x = 0$: $\quad 32 = A(2)(4) + B(-2)(4) + D(-4) = 16 - 4D \quad \Rightarrow \quad D = -4$

$x = 1$: $\quad 32 = A(3)(5) + B(-1)(5) + (C+D)(-3) = 20 - 3(C-4) \quad \Rightarrow \quad C = 0$

$$\frac{32}{(x-2)(x+2)(x^2+4)} = \frac{1}{x-2} + \frac{-1}{x+2} + \frac{-4}{x^2+4}$$

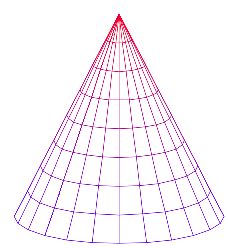
$$\int \frac{32}{(x-2)(x+2)(x^2+4)}\,dx = \int \frac{1}{x-2} + \frac{-1}{x+2} + \frac{-4}{x^2+4}\,dx \qquad x = 2\tan\theta$$

$$= \ln(x-2) - \ln(x+2) - 4\int \frac{2\sec^2\theta}{4\tan^2\theta + 4}\,d\theta = \ln(x-2) - \ln(x+2) - \int 2\,d\theta$$

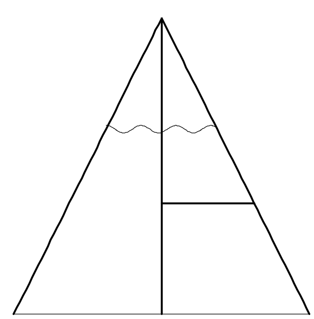
$$= \ln(x-2) - \ln(x+2) - 2\theta + C = \ln(x-2) - \ln(x+2) - 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

14. A water tank has the shape of a cone with the vertex at the top. The cone is 8 m tall and has a base which is a circle of radius 4 m. The cone is filled with water to a depth of 5 m. How much work is needed to pump the water out the top of the tank?

(The density of water is $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ and the acceleration of gravity is $g = 9.8\frac{\text{m}}{\text{sec}^2}$, but you may leave your answer as a multiple of ρg.)



Measure y down from the top. There is water where $3 \le y \le 8$ and the water at height y must be lifted a distance $D = y$. The water at height y is a circular disk of thickness dy and radius r satisfying $\frac{r}{y} = \frac{4}{8}$ or $r = \frac{1}{2}y$. So its volume is $dV = \pi r^2\,dy = \frac{\pi y^2}{4}\,dy$ and its weight is $dF = \rho g\,dV$. Therefore the work is



$$W = \int D\,dF = \int_3^8 y\rho g\frac{\pi y^2}{4}\,dy = \rho g\frac{\pi}{4}\int_3^8 y^3\,dy = \rho g\frac{\pi}{4}\left[\frac{y^4}{4}\right]_3^8 = \rho g\frac{\pi}{16}(8^4 - 3^4)$$

15. Determine if each of the following series converges or diverges. Say why.
Be sure to name or quote the test(s) you use and check out all requirements of the test.
If it converges, find the sum. **If it diverges**, does it diverge to $+\infty$, $-\infty$ or neither?

a. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ Circle one: Converges **[Diverges]**

Explain:

Apply Integral Test:

$\frac{1}{n\sqrt{\ln n}}$ is positive and decreasing Let $u = \ln n$ $du = \frac{1}{n}\,dn$

$$\int_2^{\infty} \frac{1}{n\sqrt{\ln n}}\,dn = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} = 2\sqrt{\ln n}\,\Big|_2^{\infty} = \infty$$

So $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges to $+\infty$

b. $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$ Circle one: **[Converges]** Diverges

Explain:

Apply Alternating Series Test: (Unnecessary if you get the sum.)

$(-1)^n$ is alternating. $\frac{3^n}{n!}$ is positive and decreasing for $n > 3$. $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

So $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$ converges.

OR

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges for all x.

So $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!} = e^{-3}$

16. Find the arclength of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ for $1 \le y \le 3$.

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + y^4 - 2y^2\frac{1}{4y^2} + \frac{1}{16y^4} = y^4 + \frac{1}{2} + \frac{1}{16y^4} = \left(y^2 + \frac{1}{4y^2}\right)^2$$

$$L = \int_1^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right)dy = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^3 = \left[9 - \frac{1}{12}\right] - \left[\frac{1}{3} - \frac{1}{4}\right] = \frac{53}{6}$$