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MATH 152 Final Exam Spring 2002

Sections 513,514 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. $\sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} =$

- **a.** 4
- **b.** 9 CORRECT
- **c.** $\frac{9}{7}$
- **d.** 3
- **e.** Diverges

$a = \frac{3^2}{4^{2-1}} = \frac{9}{4} \qquad r = \frac{3}{4} \qquad |r| < 1 \qquad \sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{9}{4-3} = 9$

2. Find the angle between the vectors $\vec{u} = \langle 1, 1, -1 \rangle$ and $\vec{v} = \langle 1, -2, -1 \rangle$.

- **a.** $0°$
- **b.** $30°$
- **c.** $45°$
- **d.** $60°$
- **e.** $90°$ CORRECT

$\vec{u} \cdot \vec{v} = 1 - 2 + 1 = 0 \qquad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = 0 \qquad \theta = 90°$

3. Find the arc length of the curve $x = 3\cos^2 t \qquad y = 4\sin^2 t$ for $0 \le t \le \frac{\pi}{4}$.

HINT: When you differentiate, remember the chain rule.

- **a.** $\frac{5}{2}$ CORRECT
- **b.** 5
- **c.** 6
- **d.** 12
- **e.** 10π

$\frac{dx}{dt} = -6\cos t \sin t \qquad \frac{dy}{dt} = 8\sin t \cos t$

$$L = \int_0^{\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/4} \sqrt{36\cos^2 t \sin^2 t + 64\sin^2 t \cos^2 t}\, dt = \int_0^{\pi/4} 10\sin t \cos t\, dt$$

$$= 5\sin^2 t \Big|_0^{\pi/4} = \frac{5}{2}$$

4. Find an integrating factor for the differential equation $\frac{dy}{dx} = 2xy + \sin x$.

a. $e^{-\cos x}$
b. $e^{-\sin x}$
c. $e^{\cos x}$
d. e^{x^2}
e. e^{-x^2} CORRECT

$$\frac{dy}{dx} - 2xy = \sin x \quad \Rightarrow \quad P = -2x \quad \Rightarrow \quad I = e^{\int P\,dx} = e^{-x^2}$$

5. The area between the curves $y = x^2 + 2$ and $y = 2x + 5$ for $0 \le x \le 6$ is given by the integral:

a. $\int_0^6 (x^2+2) - (2x+5)\,dx$
b. $\int_0^6 (2x+5) - (x^2+2)\,dx$
c. $\int_0^2 (x^2+2) - (2x+5)\,dx + \int_2^6 (2x+5) - (x^2+2)\,dx$
d. $\int_0^2 (2x+5) - (x^2+2)\,dx + \int_2^6 (x^2+2) - (2x+5)\,dx$
e. $\int_0^3 (2x+5) - (x^2+2)\,dx + \int_3^6 (x^2+2) - (2x+5)\,dx$ CORRECT

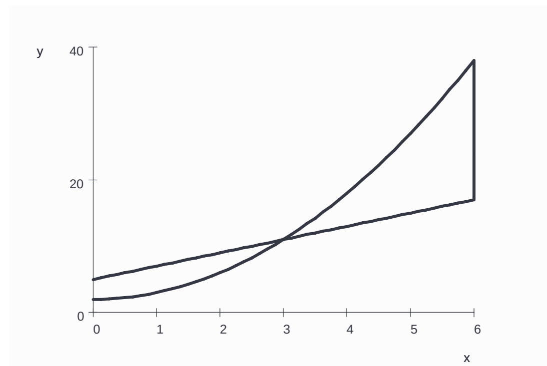
$$x^2 + 2 = 2x + 5 \quad \Rightarrow \quad x^2 - 2x - 3 = 0$$

$$\Rightarrow \quad (x+1)(x-3) = 0 \quad \Rightarrow \quad x = -1, 3$$

For $0 \le x \le 3,\quad 2x+5 \ge x^2+2$

For $3 \le x \le 6,\quad x^2+2 \ge 2x+5$

$$A = \int_0^3 (2x+5) - (x^2+2)\,dx + \int_3^6 (x^2+2) - (2x+5)\,dx$$



6. If $\vec{u}$ points North and $\vec{v}$ points South-East, then $\vec{u} \times \vec{v}$ points

a. Up
b. Down CORRECT
c. East-North-East
d. West-South-West
e. North-West

Hold your fingers North with your palm facing East. Rotate your fingers East and then South, until they point South-East. While your fingers rotate, your thumb points Down.

7. $\int_1^e 9x^2 \ln x\,dx =$

a. $2e^3 + 1$ CORRECT
b. $2e^3 - 2$
c. $2e^3$
d. $3e^3 - 3e^2$
e. $3e^3 - 3e^2 + 3$

$u = \ln x \qquad dv = 9x^2\,dx$

$du = \frac{1}{x}\,dx \qquad v = 3x^3$

$$\int_1^e 3x^2 \ln x\,dx = 3x^3 \ln x - \int 3x^2\,dx = \left[3x^3 \ln x - x^3\right]_1^e = 2e^3 + 1$$

8. The region bounded by $x = 0, \quad x = \cos y, \quad y = 0, \quad y = \frac{\pi}{4}$ is rotated about the x-axis. Which integral gives the volume of the solid of revolution?

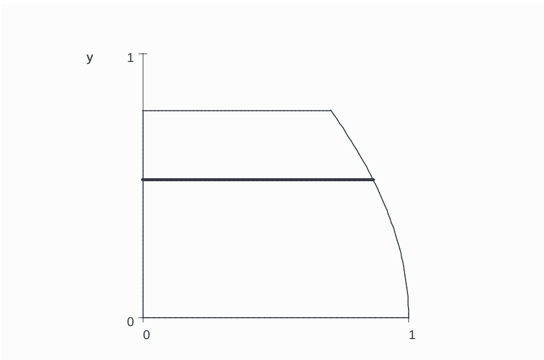
a. $\int_0^{\pi/4} 2\pi \cos^2 y\, dy$

b. $\int_0^{\sqrt{2}/2} 2\pi x \arccos x\, dx$

c. $\int_0^{\pi/4} 2\pi y \cos y\, dy$ CORRECT

d. $\int_0^{\sqrt{2}/2} \pi(\cos^2 x - x^2)\, dx$

e. $\int_0^{\pi/4} 2\pi y^2\, dy$



y-integral cylinders

$r = y \qquad h = x = \cos y$

$$V = \int_0^{\pi/4} 2\pi r h\, dy = \int_0^{\pi/4} 2\pi y \cos y\, dy$$

9. With error $|E| < 0.0001$, evaluate $\int_0^{0.1} \sin(x^2)\, dx$. HINT: Use a Maclaurin series.

a. $0.1 - \frac{(0.1)^3}{6}$

b. 0.1

c. $\frac{(0.1)^3}{3}$ CORRECT

d. $(0.1)^2 - \frac{(0.1)^6}{6}$

e. $(0.1)^2$

$\sin x = x - \frac{x^3}{6} + \cdots \qquad \sin x^2 = x^2 - \frac{x^6}{6} + \cdots \qquad \int_0^x \sin(x^2)\, dx = \frac{x^3}{3} - \frac{x^7}{42} + \cdots$

Using 1 term: $\int_0^{0.1} \sin(x^2)\, dx \approx \frac{(0.1)^3}{3}$. The error is less than the next term $|E| < \frac{(0.1)^7}{42} < 10^{-7}$

10. Using a trig substitution, $\int \frac{dx}{\sqrt{9 + 16x^2}}$ becomes

a. $\frac{1}{3} \int \cos\theta\, d\theta$

b. $\frac{1}{3} \int \tan\theta\, d\theta$

c. $\frac{3}{4} \int \sec^2\theta\, d\theta$

d. $\frac{1}{4} \int \sec\theta\, d\theta$ CORRECT

e. $\frac{1}{4} \int \sin\theta\, d\theta$

$4x = 3\tan\theta \qquad 4dx = 3\sec^2\theta\, d\theta \qquad \int \frac{dx}{\sqrt{9+16x^2}} = \int \frac{\frac{3}{4}\sec^2\theta\, d\theta}{\sqrt{9 + 9\tan^2\theta}} = \frac{3}{4} \int \frac{\sec^2\theta\, d\theta}{3\sec\theta} = \frac{1}{4} \int \sec\theta\, d\theta$

11. The partial fraction decomposition of $\dfrac{1}{x^2-x}$ is

a. $\dfrac{1}{x-1}+\dfrac{1}{x}$

b. $\dfrac{1}{x-1}-\dfrac{1}{x}$ CORRECT

c. $\dfrac{1}{x}-\dfrac{1}{x-1}$

d. $\dfrac{1}{x}+\dfrac{1}{x+1}$

e. $\dfrac{1}{x+1}-\dfrac{1}{x}$

$$\frac{1}{x^2-x}=\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}\quad\Rightarrow\quad 1=A(x-1)+Bx=(A+B)x-A$$

$$\Rightarrow\quad A+B=0,\qquad -A=1\quad\Rightarrow\quad A=-1\qquad B=1\quad\Rightarrow\quad\frac{1}{x^2-x}=\frac{-1}{x}+\frac{1}{x-1}$$

12. A 4 cm bar has density $\rho=1+5x^3\ \dfrac{\text{gm}}{\text{cm}}$ where x is measured from one end. Find its center of mass.

a. $\bar{x}=324$

b. $\bar{x}=1032$

c. $\bar{x}=\dfrac{27}{86}$

d. $\bar{x}=\dfrac{86}{27}$ CORRECT

e. $\bar{x}=\dfrac{321}{4}$

$$M=\int_0^4(1+5x^3)\,dx=\left[x+\frac{5x^4}{4}\right]_0^4=4+320=324$$

$$M_1=\int_0^4 x(1+5x^3)\,dx=\left[\frac{x^2}{2}+x^5\right]_0^4=8+1024=1032\qquad\bar{x}=\frac{M_1}{M}=\frac{1032}{324}=\frac{86}{27}\approx 3.2$$

13. $\displaystyle\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}=$

a. $\dfrac{1}{6}$

b. $\dfrac{1}{3}$ CORRECT

c. $\dfrac{1}{2}$

d. $\dfrac{2}{3}$

e. ∞

$$\sin x=x-\frac{x^3}{3!}+\cdots=x-\frac{x^3}{6}+\cdots\qquad\cos x=1-\frac{x^2}{2}+\cdots\qquad x\cos x=x-\frac{x^3}{2}+\cdots$$

$$\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+\cdots\right)-\left(x-\frac{x^3}{2}+\cdots\right)}{x^3}=\lim_{x\to 0}\frac{-\frac{x^3}{6}+\frac{x^3}{2}+\cdots}{x^3}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$$

14. If $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and the initial condition $y(0) = 3$, find $y(4)$.

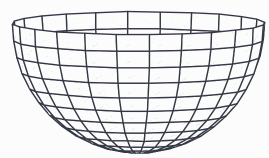
a. 1
b. 2
c. 3
d. 4
e. 5 CORRECT

$$\int y\,dy = \int x\,dx \quad \Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \pm\sqrt{x^2 + 2C}$$

$$x = 0 \ \ \& \ \ y = 3 \quad \Rightarrow \quad \frac{9}{2} = \frac{0}{2} + C \quad \Rightarrow \quad C = \frac{9}{2} \quad \Rightarrow \quad y = \sqrt{x^2 + 9} \quad \Rightarrow \quad y(4) = 5$$

Work Out Problems: (10 points each)

15. Find the work done to pump the water out the top of a hemispherical bowl of radius 5 cm if it is filled to the top.
The density of water is $\rho = 1\ \frac{\text{gm}}{\text{cm}^3}$.
The acceleration of gravity is $g = 980\ \frac{\text{cm}}{\text{sec}^2}$.



Measure y down from the top. The slice at depth y must be lifted a distance $D = y$, and has radius r which satisfies $r^2 + y^2 = 25$. So this slice has volume $dV = \pi r^2\,dy = \pi(25 - y^2)\,dy$. The force to lift this slice is its weight $dF = \rho g\,dV = \rho g\pi(25 - y^2)\,dy$. So the work is

$$W = \int_0^5 D\,dF = \int_0^5 y\rho g\pi(25 - y^2)\,dy = \rho g\pi \int_0^5 (25y - y^3)\,dy = \rho g\pi\left[25\frac{y^2}{2} - \frac{y^4}{4}\right]_0^5$$

$$= \rho g\pi 5^4\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{\rho g\pi 5^4}{4} = \frac{980\pi 625}{4}$$

16. Find the radius and interval of convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{(x-4)^n}{3\ln n}$. Be sure to check the endpoints.

Ratio Test: $\quad a_n = \dfrac{(x-4)^n}{3\ln n} \qquad a_{n+1} = \dfrac{(x-4)^{n+1}}{3\ln(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{3\ln(n+1)}\frac{3\ln n}{(x-4)^n}\right| = |x-4|\lim_{n\to\infty}\left|\frac{\ln n}{\ln(n+1)}\right| = |x-4|\lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = |x-4| < 1$$

Radius of convergence is $R = 1$.

Converges if $3 < x < 5$.

At $x = 3$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{3\ln n}$ which converges by the Alternating Series Test.

At $x = 5$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3\ln n}$. We apply the Comparison Test with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3n}$ which is a divergent harmonic series. Since $n > \ln n$ we have $\dfrac{1}{3n} < \dfrac{1}{3\ln n}$ and hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3\ln n}$ also diverges.

So the interval of convergence is $[3,5)$.

17. Determine if the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ converges absolutely, converges conditionally or diverges.

If it converges, find the sum. If it diverges, does it diverge to $+\infty$, $-\infty$ or neither?

Show all work and name any tests you use.

Circle One: **[Converges Absolutely]** Converges Conditionally Diverges

Fill in the Blank: Converges to _________e^{-2}_________

Or Circle One: Diverges to $+\infty$ $-\infty$ Neither

The related absolute series is $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$. Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we conclude $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

Alternatively, apply the ratio test: $\quad a_n = \dfrac{2^n}{n!} \qquad a_{n+1} = \dfrac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{(n+1)} = 0 < 1$$

$\Rightarrow$ Absolute series converges and original series converges absolutely.

Similarly, since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we conclude $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2}$.

$\Rightarrow$ Original series converges to e^{-2}.