MATH 152 FINAL EXAM Spring 2005

Sections 813-815 Version A Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Find the average value of $f(x) = \cos x$ on the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

a. $\frac{2\sqrt{2}}{\pi}$ CORRECT

b. $\frac{\sqrt{2}}{\pi}$

c. $\sqrt{2}$

d. $\frac{1}{\sqrt{2}}$

e. $\frac{\pi}{\sqrt{2}}$

$$f_{\text{ave}} = \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} \cos x\,dx = \frac{2}{\pi}\Big[\sin x\Big]_{-\pi/4}^{\pi/4} = \frac{2}{\pi}\left(\frac{1}{\sqrt{2}}\right) - \frac{2}{\pi}\left(-\frac{1}{\sqrt{2}}\right) = \frac{4}{\pi\sqrt{2}} = \frac{2\sqrt{2}}{\pi}$$

2. The ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$ is revolved about the x-axis. Which integral gives the volume of the resulting ellipsoid?

a. $\int_{-2}^{2} 2\pi x\sqrt{16-4x^2}\,dx$

b. $\int_{-4}^{4} 2\pi(16-4x^2)^2\,dx$

c. $\int_{-2}^{2} \pi(16-4x^2)\,dx$ CORRECT

d. $\int_{-4}^{4} 2\pi x\sqrt{16-4x^2}\,dx$

e. $\int_{-2}^{2} \pi(16-4x^2)^2\,dx$

x integral, disks $V = \int_{-2}^{2} \pi y^2\,dx = \int_{-2}^{2} \pi(16-4x^2)\,dx$

3. Compute $\int_{0}^{\pi/4} \cos\theta\sin^3\theta\,d\theta$.

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{1}{8}$

d. $\frac{1}{16}$ CORRECT

e. $\frac{1}{32}$

$u = \sin\theta \qquad du = \cos\theta\,d\theta$

$$\int_{0}^{\pi/4} \cos\theta\sin^3\theta\,dx = \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/4} = \frac{1}{4}\frac{1}{\sqrt{2}^{\,4}} = \frac{1}{16}$$

4. Compute $\int_0^{\ln 2} xe^{-x}\,dx$.

a. $\frac{1}{2} + \frac{1}{2}\ln 2$

b. $\frac{1}{2} - \frac{1}{2}\ln 2$ CORRECT

c. $\frac{1}{2}\ln 2 - \frac{1}{2}$

d. $-\frac{1}{2}\ln 2 - \frac{1}{2}$

e. Divergent

Parts: $u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^{\ln 2} xe^{-x}\,dx = \left[-xe^{-x} + \int e^{-x}\,dx\right]_0^{\ln 2} = \left[-xe^{-x} - e^{-x}\right]_0^{\ln 2} = (-\ln 2e^{-\ln 2} - e^{-\ln 2}) - (-1) = -\frac{1}{2}\ln 2 + \frac{1}{2}$$

5. Use the Trapezoid Rule with $n = 4$ intervals to approximate the integral $\int_1^9 (9 + x^2)\,dx$.

a. 240

b. 312

c. $314\frac{1}{3}$

d. 320 CORRECT

e. 400

$\Delta x = \frac{9-1}{4} = 2$

$$T_4 = \Delta x\left(\frac{1}{2}f(1) + f(3) + f(5) + f(7) + \frac{1}{2}f(9)\right) = 2\left(\frac{1}{2}10 + 18 + 34 + 58 + \frac{1}{2}90\right) = 320$$

6. A barrel initially contains 3 cups of sugar dissolved in 4 gallons of water. You then add pure water at the rate of 2 gallons per minute while the mixture is draining out of a hole in the bottom at 2 gallons per minute. Find the amount of sugar in the barrel after 2 minute.

a. $\frac{3}{\sqrt{e}}$

b. $\frac{3}{e}$ CORRECT

c. $3e$

d. $3\sqrt{e}$

e. $\frac{3}{e^2}$

Let $S(t)$ be the cups of sugar at time t.

$$\frac{dS}{dt}\frac{\text{cups}}{\text{min}} = -2\frac{\text{gal}}{\text{min}}\frac{S\text{ cups}}{4\text{ gal}} \qquad \frac{dS}{dt} = -\frac{1}{2}S \qquad S(0) = 3$$

$$S(t) = 3e^{-t/2} \qquad S(2) = 3e^{-1} = \frac{3}{e}$$

7. As n approaches infinity, the sequence $\left\{\frac{1-\cos n}{n^2}\right\}$

a. converges to $-\frac{1}{2}$

b. converges to 0 CORRECT

c. converges to $\frac{1}{2}$

d. converges to 1

e. diverges

Since $0 \le \frac{1-\cos n}{n^2} \le \frac{2}{n^2}$ and $\lim_{n\to\infty} \frac{2}{n^2} = 0,$ the Squeeze Theorem says $\lim_{n\to\infty} \frac{1-\cos n}{n^2} = 0.$

8. Compute $\sum_{n=1}^{\infty}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right)$

a. $-\frac{1}{2}$ CORRECT

b. $\frac{1}{2}$

c. 1

d. 2

e. Divergent

$$S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{k}{k+1} - \frac{k+1}{k+2}\right) = \frac{1}{2} - \frac{k+1}{k+2}$$

$$S = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{k+1}{k+2}\right) = -\frac{1}{2}$$

9. Find the 4^{th} degree Taylor polynomial for $f(x) = x^2 - x$ about $x = 2$.

a. $T_4(x) = 2 + 3(x-2) + (x-2)^2 + 3(x-2)^3 + (x-2)^4$

b. $T_4(x) = 2 + 3(x-2) + 2(x-2)^2 + 3(x-2)^3 + 2(x-2)^4$

c. $T_4(x) = 2 + 3(x-2) + (x-2)^2$ CORRECT

d. $T_4(x) = 2 + 3(x-2) + 2(x-2)^2$

e. $T_4(x)$ cannot be found because $x = 2$ is outside the interval of convergence.

$f(x) = x^2 - x$ $\quad f'(x) = 2x - 1$ $\quad f''(x) = 2$ $\quad f^{(n)}(x) = 0$ for $n \ge 3$.

$f(2) = 2$ $\quad f'(2) = 3$ $\quad f''(2) = 2$ $\quad f^{(n)}(2) = 0$ for $n \ge 3$.

$$T_4(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 = 2 + 3(x-2) + (x-2)^2$$

10. A triangle has vertices $A = (0,3,2)$, $B = (-2,3,0)$ and $C = (-2,0,3)$.
Find the angle at vertex B.

a. $\frac{\pi}{6}$
b. $\frac{\pi}{3}$ CORRECT
c. $\frac{\pi}{2}$
d. $\frac{2\pi}{3}$
e. $\frac{5\pi}{6}$

$$\overrightarrow{BA} = A - B = (2,0,2) \qquad \overrightarrow{BC} = C - B = (0,-3,3) \qquad \overrightarrow{BA} \cdot \overrightarrow{BC} = 6$$

$$\left|\overrightarrow{BA}\right| = \sqrt{4+4} = 2\sqrt{2} \qquad \left|\overrightarrow{BC}\right| = \sqrt{9+9} = 3\sqrt{2}$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{6}{2\sqrt{2}\,3\sqrt{2}} = \frac{1}{2} \qquad \Rightarrow \qquad \theta = 60^\circ = \frac{\pi}{3}$$

11. If $\vec{u}$ points South-West and $\vec{v}$ points Up, which way does $\vec{u} \times \vec{v}$ point?

a. South-East
b. North-East
c. North-West CORRECT
d. 45° Up from North-West
e. 45° Down from North-West

Hold your right fingers South-West with the palm facing Up. Then your thumb points North-West.

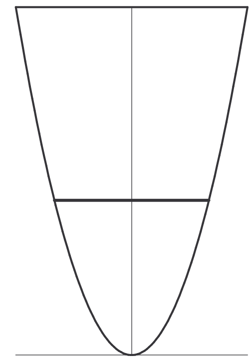
12. Find the area of a triangle with edges $\vec{a} = (3,-2,1)$ and $\vec{b} = (-1,0,1)$.

a. 1
b. 2
c. $\sqrt{6}$ CORRECT
d. 6
e. $2\sqrt{6}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{\imath}(-2-0) - \vec{\jmath}(3+1) + \vec{k}(0-2) = (-2,-4,-2)$$

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{4+16+4} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

13. (12 points) The end of a water trough occupies the region between $y = x^2$ m and $y = 9$ m.
It is filled to a depth of $y = 4$ m.
Find the force on the end of the trough.
Give your answer in terms of ρ (the density of water) and g (the acceleration of gravity).



There is only water between $y = 0$ and $y = 4$.

The slice at height y is $h = 4 - y$ below the surface and has area

$$dA = w\,dy = 2x\,dy = 2\sqrt{y}\,dy.$$

So the force is

$$F = \int_0^4 P\,dA = \int_0^4 \rho g h w\,dy = \rho g \int_0^4 (4-y)2\sqrt{y}\,dy = 2\rho g \int_0^4 (4y^{1/2} - y^{3/2})\,dy$$

$$= 2\rho g\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 = 2\rho g\left(\frac{64}{3} - \frac{64}{5}\right) = 128\rho g\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{256}{15}\rho g$$

14. (12 points) Compute $\displaystyle\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx$.

$x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$

$$\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx = \int_0^{\pi/4} \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta = 3\int_0^{\pi/4} \tan^2\theta\,d\theta$$

$$= 3\int_0^{\pi/4} \sec^2\theta - 1\,d\theta = 3\left[\tan\theta - \theta\right]_0^{\pi/4} = 3\left(1 - \frac{\pi}{4}\right)$$

15. (12 points) Find the arc length of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ between $x = 1$ and $x = e$.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \left(\frac{e^2}{4} + \frac{1}{2}\right) - \left(\frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

16. (12 points) The Taylor series $f(x) = \sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^{n-1}$ is obtained by differentiating the series $g(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$. What is the the function $f(x)$? What is its interval of convergence for $f(x)$ (including endpoints)? Justify your answers.

$g(x)$ is a geometric series with ratio $r = \frac{x-1}{2}$ and first term $a = 1$.

So $g(x) = \frac{1}{1 - \left(\frac{x-1}{2}\right)} = \frac{2}{3-x}$ and it converges for $\left|\frac{x-1}{2}\right| < 1$ or $|x-1| < 2$.

So the center is $x = 1$ and the radius of convergence is $R = 2$.

Then $f(x) = g'(x) = \frac{2}{(3-x)^2}$ with center $x = 1$ and radius of convergence $R = 2$.

The interval of convergence is $(-1, 3)$ except we need to check the endpoints.

At $x = -1$: $f(-1) = \sum_{n=1}^{\infty} \frac{n}{2^n}(-2)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{2}$ which diverges by the n^{th} Term Divergence Test.

At $x = 3$: $f(3) = \sum_{n=1}^{\infty} \frac{n}{2^n}(2)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2}$ which diverges by the n^{th} Term Divergence Test.

So the interval of convergence is $(-1, 3)$.