Spring 2005 Math 152
Exam 3A: Solutions
Mon, 02/May ©2005, Art Belmonte

1. (e)

- Examine the corresponding *series*.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-10)^n}{n!} = e^{-10}$$

Since this series converges, we *must* have $\lim_{n\to\infty} a_n = 0$.

- *Alternatively,* look at $\ln|a_n|$. As $n \to \infty$, we have

$$\ln \frac{10^n}{n!} = n \ln 10 - \sum_{k=1}^{n} \ln k = \sum_{k=1}^{n} (\ln 10 - \ln k) \to -\infty.$$

Thus $\lim |a_n| = \lim e^{\ln|a_n|} = 0$. Hence $\lim a_n = 0$.

2. (c) The series $\sum (-1)^n e^{1/n}$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n e^{1/n} \neq 0$. Indeed, $\liminf a_n = -1$ and $\limsup a_n = +1$.

3. (d) Compute a few partial sums of this *telescoping* series until it's clear what's happening. Now $s_1 = \cos\frac{1}{2} - \cos\frac{1}{3}$, $s_2 = \cos\frac{1}{2} - \cos\frac{1}{4}$, $s_3 = \cos\frac{1}{2} - \cos\frac{1}{5}$, and in general, $s_n = \cos\frac{1}{2} - \cos\frac{1}{n+2}$. Hence as $n \to \infty$, we have $s_n \to \cos\frac{1}{2} - \cos 0 = \cos\frac{1}{2} - 1$.

4. (d) This series converges via the Geometric Series Theorem.

$$\sum_{n=1}^{\infty} \frac{3\,(2)^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3\,(4)}{5^2}\left(\frac{4}{5}\right)^{n-1} = \frac{12/25}{1-\frac{4}{5}} = \frac{12/25}{1/5} = \frac{12}{5}$$

5. (b) For all real x we have

$$x\cos\left(x^3\right) = x\sum_{n=0}^{\infty} \frac{(-1)^n \left(x^3\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}.$$

6. (c) Since the power series $\sum c_n x^n$, centered at $a = 0$, converges at $x = 3$ and diverges at $x = 5$, we know that the radius of convergence R is at least 3 and at most 5. Accordingly, it *must* be true that series converges at $x = 2$, but diverges at $x = 6$. (For $x = 4$, the series may converge or it may diverge.)

7. (d) At $x = \frac{\pi}{3}$ we have

$$\begin{aligned} f(x) &= \cos x = 1/2 \\ f'(x) &= -\sin x = -\sqrt{3}/2 \\ f''(x) &= -\cos x = -1/2 \\ f'''(x) &= \sin x = \sqrt{3}/2 \end{aligned}$$

Therefore,

$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}\left(x - \frac{\pi}{3}\right)^n \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 \end{aligned}$$

8. (e) The desired coefficient is $c_3 = \dfrac{f'''(3)}{3!}$. Compute the requisite derivatives of $f(x) = \ln x$: $f'(x) = 1/x = x^{-1}$, $f''(x) = -x^{-2}$, and $f'''(x) = 2x^{-3}$. So $c_3 = \dfrac{2/27}{6} = \dfrac{1}{81}$.

9. (c) We have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n-5}{2+n} = \lim_{n\to\infty} \frac{4 - \frac{5}{n}}{\frac{2}{n} + 1} = 4.$$

10. (e) Since $f(x) = 1/x = x^{-1}$, we have $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, and $f'''(x) = -6x^{-4}$. Thus for $2 \le x \le 6$, $\left|f^{(3)}(x)\right| = \dfrac{6}{x^4} \le \dfrac{6}{(2)^4} = M$ and therefore $|R_2(x)| \le \dfrac{M\,|x-4|^3}{3!} \le \dfrac{\left(6/2^4\right)(2)^3}{6} = \dfrac{1}{2}$ for $2 \le x \le 6$.

11. Use the Ratio Test or the Root Test with GFF to determine the radius of convergence R.

- The series will converge via the Ratio Test provided

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{2^{n+1}\,|x+1|^{n+1}}{n+1} \cdot \frac{n}{2^n\,|x+1|^n} \\ &= \lim_{n\to\infty} \frac{2\,|x+1|}{1+\frac{1}{n}} = 2\,|x+1| < 1 \end{aligned}$$

or $|x - (-1)| < \frac{1}{2}$. Thus $R = \frac{1}{2}$.

- Or, as $n \to \infty$ the Root Test with GFF requires

$$\sqrt[n]{|a_n|} = \frac{2\,|x+1|}{\sqrt[n]{n}} \to 2\,|x+1| < 1$$

or $|x - (-1)| < \frac{1}{2}$. Thus $R = \frac{1}{2}$.

- With center $a = -1$ and radius $R = \frac{1}{2}$, let's examine convergence of the series at the endpoints of the interval $(a - R, a + R) = \left(-\frac{3}{2}, -\frac{1}{2}\right)$. At $x = -\frac{3}{2}$, the series is $\sum \frac{1}{n}$, the divergent harmonic series (or p-series with $p = 1 \le 1$). At $x = -\frac{1}{2}$, the we have the alternating harmonic series $\sum \frac{(-1)^n}{n}$, which converges by the Alternating Series Test since $b_n = |a_n| = \frac{1}{n} \downarrow 0$. Hence the interval of convergence is $I = \left(-\frac{3}{2}, -\frac{1}{2}\right]$.

[Please turn the page for solutions to Problems 12–15.]

12.
- A series $\sum a_n$ converges absolutely if and only if the series of absolute values $\sum |a_n|$ converges.
- Accordingly, the series in question converges absolutely via the Integral Test.

$$\begin{aligned}\int_2^\infty (\ln x)^{-4} \frac{1}{x}\, dx &= \lim_{t\to\infty} \left(-\tfrac{1}{3}(\ln x)^{-3}\right)\Big|_2^t \\ &= \lim_{t\to\infty}\left(-\tfrac{1}{3(\ln t)^3} + \tfrac{1}{3(\ln 2)^3}\right) \\ &= \frac{1}{3(\ln 2)^3}\end{aligned}$$

13. (a) We have

$$\begin{aligned}\int_0^{0.1} e^{-x^2}\, dx &= \int_0^{1/10} \sum_{n=0}^\infty \frac{\left(-x^2\right)^n}{n!}\, dx \\ &= \int_0^{1/10} \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{n!}\, dx \\ &= \left(\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}\right)\Bigg|_0^{1/10} \\ &= \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{10}\right)^{2n+1}}{(2n+1)\, n!}\end{aligned}$$

(b) The third partial sum is $\frac{1}{10} - \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{1}{10}\right)^5$ or approximately 0.099668.

(c) The Alternating Series Estimation Theorem guarantees that the magnitude of the error in this approximation is less than or equal to that of the first neglected term. This corresponds to $n = 3$. Therefore, the error satisfies $|\text{error}| \le \dfrac{10^{-7}}{7\,(3!)} = \dfrac{1}{42 \times 10^7} \approx 2.38 \times 10^{-9}$.

14. (a) The series $\sum \dfrac{(-1)^n}{n^{3/4}}$ converges via the Alternating Series Test since $b_n = |a_n| = \dfrac{1}{n^{3/4}} \downarrow 0$.

(b) The series $\sum \dfrac{n^2}{n^4 - n}$ is asymptotically similar to the convergent p-series $\sum \dfrac{1}{n^2}$ (here $p = 2 > 1$) and thus converges by the Limit Comparison Theorem.

$$\lim_{n\to\infty} \frac{\dfrac{n^2}{n^4-n}}{1/n^2} = \lim_{n\to\infty} \frac{n^4}{n^4-n} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{n^3}} = 1 > 0$$

15.
- Computing the Maclaurin series via the definition is straightforward. ("Brute force has a charm all its own.")

$$\begin{aligned} f(x) &= \ln(1+8x) \\ f'(x) &= 8(1+8x)^{-1} \\ f''(x) &= -8^2(1+8x)^{-2} \\ f'''(x) &= 2\cdot 8^3 (1+8x)^{-3} \\ f^{(4)}(x) &= -6\cdot 8^4(1+8x)^{-4} \\ &\vdots \\ f^{(n)}(x) &= (-1)^{n-1}(n-1)!\cdot 8^n(1+8x)^{-n}\end{aligned}$$

Thus $f^{(n)}(0) = (-1)^{n-1} 8^n (n-1)!$ for $n \ge 1$ and $f^{(0)}(0) = f(0) = \ln 1 = 0$.

- Hence
$\ln(1+8x) = f(x) = \displaystyle\sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^\infty \frac{(-1)^{n-1} 8^n x^n}{n}$.
- As $n \to \infty$ the Root Test with GFF requires

$$\sqrt[n]{|a_n|} = \frac{8|x|}{\sqrt[n]{n}} \to 8|x| < 1$$

or $|x| < \frac{1}{8}$. Thus $R = \frac{1}{8}$. (The Ratio Test gives the same result.)

- *Alternatively,* manipulate a known geometric series. Note that $\ln(1+z)$ is an antiderivative of $\dfrac{1}{1+z}$.

$$\begin{aligned}\ln(1+z) &= \int \frac{1}{1+z}\, dz = \int \frac{1}{1-(-z)}\, dz \\ &= \int \sum_{n=0}^\infty (-z)^n\, dz, \quad \text{if } |-z| < 1 \\ &= \int \sum_{n=0}^\infty (-1)^n z^n\, dz, \quad \text{if } |z| < 1 \\ &= C + \sum_{n=0}^\infty (-1)^n \frac{z^{n+1}}{n+1} \\ \ln(1+z) &= C + \sum_{k=1}^\infty (-1)^{k-1} \frac{z^k}{k} \\ 0 = \ln(1+0) &= C + \sum 0 = C \\ \text{Thus } \ln(1+z) &= \sum_{k=1}^\infty \frac{(-1)^{k-1} z^k}{k}.\end{aligned}$$

We require $|z| < 1$. Hence the radius of convergence for *this* series is $R = 1$.

- Now set $z = 8x$. Then
$\ln(1+8x) = \displaystyle\sum_{k=1}^\infty \frac{(-1)^{k-1}(8x)^k}{k} = \sum_{k=1}^\infty \frac{(-1)^{k-1} 8^k x^k}{k}$
provided $|z| = |8x| < 1$ or $|x| < \frac{1}{8}$. Thus the radius of convergence of *our* series is $R = \frac{1}{8}$.