Multiple Choice: (15 questions, 4 points each)

1. Compute $\displaystyle\int_0^{\pi/4} \cos\theta \sin^3\theta\, d\theta$

   a. $\dfrac{1}{2}$

   b. $\dfrac{1}{4}$

   c. $\dfrac{1}{8}$

   d. $\dfrac{1}{16}$ CORRECT

   e. $\dfrac{1}{32}$

$$u = \sin\theta \qquad du = \cos\theta\, d\theta \qquad \int_0^{\pi/4} \cos\theta \sin^3\theta\, dx = \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/4} = \frac{1}{4}\frac{1}{\sqrt{2}^4} = \frac{1}{16}$$

2. Compute $\displaystyle\int_0^{\ln 2} xe^{-x}\, dx$

   a. $\dfrac{1}{2} + \dfrac{1}{2}\ln 2$

   b. $\dfrac{1}{2} - \dfrac{1}{2}\ln 2$ CORRECT

   c. $\dfrac{1}{2}\ln 2 - \dfrac{1}{2}$

   d. $-\dfrac{1}{2}\ln 2 - \dfrac{1}{2}$

   e. Divergent

$$\text{Parts:} \qquad \begin{array}{ll} u = x & dv = e^{-x}\, dx \\ du = dx & v = -e^{-x} \end{array}$$

$$\int_0^{\ln 2} xe^{-x}\, dx = \left[-xe^{-x} + \int e^{-x}\, dx\right]_0^{\ln 2} = \left[-xe^{-x} - e^{-x}\right]_0^{\ln 2} = (-\ln 2 e^{-\ln 2} - e^{-\ln 2}) - (-1) = -\frac{1}{2}\ln 2 + \frac{1}{2}$$

3. Find the average value of $f(x) = \cos x$ on the interval $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$.

   a. $\dfrac{2\sqrt{2}}{\pi}$ CORRECT

   b. $\dfrac{\sqrt{2}}{\pi}$

   c. $\sqrt{2}$

   d. $\dfrac{1}{\sqrt{2}}$

   e. $\dfrac{\pi}{\sqrt{2}}$

$$f_{\text{ave}} = \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} \cos x\, dx = \frac{2}{\pi}\left[\sin x\right]_{-\pi/4}^{\pi/4} = \frac{2}{\pi}\left(\frac{1}{\sqrt{2}}\right) - \frac{2}{\pi}\left(-\frac{1}{\sqrt{2}}\right) = \frac{4}{\pi\sqrt{2}} = \frac{2\sqrt{2}}{\pi}$$

4. The partial fraction decomposition of $\dfrac{1}{x^2 - x}$ is

a. $\dfrac{1}{x-1} + \dfrac{1}{x}$

b. $\dfrac{1}{x-1} - \dfrac{1}{x}$ CORRECT

c. $\dfrac{1}{x} - \dfrac{1}{x-1}$

d. $\dfrac{1}{x} + \dfrac{1}{x+1}$

e. $\dfrac{1}{x+1} - \dfrac{1}{x}$

$$\dfrac{1}{x^2 - x} = \dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} \quad \Rightarrow \quad 1 = A(x-1) + Bx = (A+B)x - A$$

$$\Rightarrow \quad A + B = 0, \quad -A = 1 \quad \Rightarrow \quad A = -1 \quad B = 1 \quad \Rightarrow \quad \dfrac{1}{x^2 - x} = \dfrac{-1}{x} + \dfrac{1}{x-1}$$

5. The base of a solid is the semi-circle between $y = \sqrt{9 - x^2}$ and the x-axis. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.
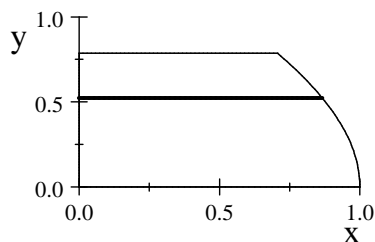
a. $\dfrac{9}{2}\pi$

b. 9π

c. 9

d. 18

e. 36 CORRECT

This is a x-integral. The side of the square is $s = \sqrt{9 - x^2}$. So the area is $A = s^2 = 9 - x^2$.

So the volume is $\quad V = \displaystyle\int_{-3}^{3} A\,dx = \int_{-3}^{3}(9 - x^2)\,dx = \left[9x - \dfrac{x^3}{3}\right]_{-3}^{3} = 2(27 - 9) = 36$

6. The region bounded by $x = 0$, $x = \cos y$, $y = 0$, $y = \dfrac{\pi}{4}$ is rotated about the x-axis. Which integral gives the volume of the solid of revolution?

a. $\displaystyle\int_{0}^{\pi/4} 2\pi \cos^2 y\,dy$

b. $\displaystyle\int_{0}^{\sqrt{2}/2} 2\pi x \arccos x\,dx$

c. $\displaystyle\int_{0}^{\pi/4} 2\pi y \cos y\,dy$ CORRECT

d. $\displaystyle\int_{0}^{\sqrt{2}/2} \pi(\cos^2 x - x^2)\,dx$

e. $\displaystyle\int_{0}^{\pi/4} 2\pi y^2\,dy$

y-integral   cylinders

$r = y \qquad h = x = \cos y$

$V = \displaystyle\int_{0}^{\pi/4} 2\pi r h\,dy = \int_{0}^{\pi/4} 2\pi y \cos y\,dy$

2

7. As n approaches infinity, the sequence $a_n = \dfrac{1 - \cos n}{n^2}$

    a. converges to $-\dfrac{1}{2}$

    b. converges to 0 CORRECT

    c. converges to $\dfrac{1}{2}$

    d. converges to 1

    e. diverges

Since $0 \le \dfrac{1 - \cos n}{n^2} \le \dfrac{2}{n^2}$ and $\lim\limits_{n \to \infty} \dfrac{2}{n^2} = 0$, the Squeeze Theorem
says $\lim\limits_{n \to \infty} \dfrac{1 - \cos n}{n^2} = 0.$

8. $\displaystyle\sum_{n=2}^{\infty} \dfrac{3^n}{4^{n-1}} =$

    a. $\dfrac{9}{7}$

    b. 4

    c. 9 CORRECT

    d. 3

    e. Diverges

$$a = \frac{3^2}{4^{2-1}} = \frac{9}{4} \qquad r = \frac{3}{4} \qquad |r| < 1 \qquad \sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} = \frac{\frac{9}{4}}{1 - \frac{3}{4}} = \frac{9}{4 - 3} = 9$$

9. Estimate $\displaystyle\int_0^{0.1} \sin(x^2)\,dx$ to within an error of $|E| < 0.0001,$ HINT: Use a Maclaurin series.

    a. $0.1 - \dfrac{(0.1)^3}{6}$

    b. 0.1

    c. $(0.1)^2 - \dfrac{(0.1)^6}{6}$

    d. $\dfrac{(0.1)^3}{3}$ CORRECT

    e. $(0.1)^2$

$$\sin x = x - \frac{x^3}{6} + \cdots \qquad \sin x^2 = x^2 - \frac{x^6}{6} + \cdots \qquad \int_0^x \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{42} + \cdots$$

Using 1 term: $\displaystyle\int_0^{0.1} \sin(x^2)\,dx \approx \dfrac{(0.1)^3}{3}.$ The error is less than the next term

$|E| < \dfrac{(0.1)^7}{42} < 10^{-7}$

10. Compute $\displaystyle\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3}$

a. $-\dfrac{1}{9}$

b. -4

c. $-\dfrac{8}{9}$

d. $-\dfrac{4}{3}$

e. $-\dfrac{4}{9}$ CORRECT

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3} = \lim_{x\to 0} \frac{\left[2x - \dfrac{(2x)^3}{3!} + \cdots\right] - 2x}{3x^3} = \lim_{x\to 0}\left[-\frac{(2x)^3}{3!3x^3} + \cdots\right] = -\frac{8}{6\cdot 3} = -\frac{4}{9}$$

11. Compute $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right)$

a. $-\dfrac{1}{2}$ CORRECT

b. $\dfrac{1}{2}$

c. 1

d. 2

e. Divergent

$$S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{k}{k+1} - \frac{k+1}{k+2}\right) = \frac{1}{2} - \frac{k+1}{k+2}$$

$$S = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{k+2}\right) = -\frac{1}{2}$$

12. If $g(x) = \cos(x^2)$, what is $g^{(8)}(0)$, the 8$^{\text{th}}$ derivative at zero?

HINT: What is the coefficient of x^8 in the Maclaurin series for $\cos(x^2)$?

a. $\dfrac{1}{8\cdot 7\cdot 6\cdot 5}$

b. $4!$

c. $\dfrac{1}{4!}$

d. $8\cdot 7\cdot 6\cdot 5$ CORRECT

e. $\dfrac{1}{8!}$

On the one hand, the Maclaurin series for $\cos(t)$ is $\cos(t) = 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \cdots$.

So the Maclaurin series for $\cos(x^2)$ is $\cos(x^2) = 1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \cdots$.

On the other hand the Maclaurin series for any function $g(x)$ is

$$g(x) = g(0) + g'(0)x + \cdots + \frac{g^{(8)}(0)}{8!}x^8 + \cdots$$

Since these must be equal, the coefficients of x^8 must be equal: $\dfrac{g^{(8)}(0)}{8!} = \dfrac{1}{4!}$

So $g^{(8)}(0) = \dfrac{8!}{4!} = 8\cdot 7\cdot 6\cdot 5 = 1680$

4

13. A vector $\vec{u}$ has length 5. A vector $\vec{v}$ has length 4. The angle between them is $60°$. Find $\vec{u} \cdot \vec{v}$.

    a. 10 CORRECT

    b. $\dfrac{1}{40}$

    c. $\dfrac{\sqrt{3}}{40}$

    d. 40

    e. $10\sqrt{3}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = 5 \cdot 4 \cdot \cos 60° = \frac{20}{2} = 10$$

14. If $\vec{u}$ points North and $\vec{v}$ points South-East, then $\vec{u} \times \vec{v}$ points

    a. Up

    b. Down   CORRECT

    c. East-North-East

    d. West-South-West

    e. North-West

Hold your right fingers North with your palm facing South-East. Your thumb points Down.

15. Find the area of a triangle with edges $\vec{a} = (3, -2, 1)$ and $\vec{b} = (-1, 0, 1)$.

    a. 1

    b. 2

    c. $\sqrt{6}$ CORRECT

    d. 6

    e. $2\sqrt{6}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i}(-2-0) - \vec{j}(3+1) + \vec{k}(0-2) = (-2, -4, -2)$$

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{4+16+4} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

Work Out (4 questions, 12 points each)

Show all you work.

16. Compute $\displaystyle\int_{3}^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx.$

$x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$

$\displaystyle\int_{3}^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx = \int_{0}^{\pi/4} \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta = 3\int_{0}^{\pi/4} \tan^2\theta\,d\theta$

$\displaystyle = 3\int_{0}^{\pi/4} \sec^2\theta - 1\,d\theta = 3\Big[\tan\theta - \theta\Big]_{0}^{\pi/4} = 3\Big(1 - \frac{\pi}{4}\Big)$

17. Find the work done to pump the water out the top of a hemispherical bowl of radius 5 cm if it is filled to the top.

The density of water is $\rho = 1\,\dfrac{\text{gm}}{\text{cm}^3}$.

The acceleration of gravity is $g = 980\,\dfrac{\text{cm}}{\text{sec}^2}$.

Measure y down from the top.
The slice at depth y must be lifted a distance $D = y$,
and has radius r which satisfies $r^2 + y^2 = 25$.
So this slice has volume $dV = \pi r^2\,dy = \pi(25 - y^2)\,dy$.
The force to lift this slice is its weight $dF = \rho g\,dV = \rho g\pi(25 - y^2)\,dy$.
So the work is

$\displaystyle W = \int_{0}^{5} D\,dF = \int_{0}^{5} y\rho g\pi(25 - y^2)\,dy = \rho g\pi\int_{0}^{5}(25y - y^3)\,dy = \rho g\pi\left[25\frac{y^2}{2} - \frac{y^4}{4}\right]_{0}^{5}$

$\displaystyle = \rho g\pi 5^4\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{\rho g\pi 5^4}{4} = \frac{980\pi 625}{4}$

18. Find the arc length of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ between $x = 1$ and $x = e$.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \left(\frac{e^2}{4} + \frac{1}{2}\right) - \left(\frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

19. Find the radius and interval of convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^n}{4\ln n}$. Be sure to check the endpoints. Name or state any test you use and check the conditions.

Ratio Test: $\quad a_n = \dfrac{(x-3)^n}{4\ln n} \qquad a_{n+1} = \dfrac{(x-3)^{n+1}}{4\ln(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{4\ln(n+1)}\frac{4\ln n}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\ln n}{\ln(n+1)}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = |x-3| < 1$$

$$\boxed{\text{Radius: } R = 1.}$$

Converges if $2 < x < 4$.

At $x = 2$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{4\ln n}$ which converges by the Alternating Series Test.

The $(-1)^n$ says it alternates. The $\dfrac{1}{4\ln n}$ decreases and $\displaystyle\lim_{n\to\infty}\frac{1}{4\ln n} = 0$.

At $x = 4$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4\ln n}$. We apply the Comparison Test with $\displaystyle\sum_{n=2}^{\infty}\frac{1}{3n}$ which is a

divergent harmonic series. Since $n > \ln n$ we have $\dfrac{1}{3n} < \dfrac{1}{3\ln n}$ and hence $\displaystyle\sum_{n=2}^{\infty}\frac{1}{3\ln n}$

also diverges.

$$\boxed{\text{Interval: } [2,4).}$$