MATH 152 FINAL EXAM Spring 2012

Sections 531-533 Version B Solutions P. Yasskin

Multiple Choice: (15 questions, 4 points each)

1. Compute $\int_0^{\ln 2} xe^{-x}\,dx$

a. $\frac{1}{2}+\frac{1}{2}\ln 2$

b. $\frac{1}{2}-\frac{1}{2}\ln 2$ CORRECT

c. $\frac{1}{2}\ln 2-\frac{1}{2}$

d. $-\frac{1}{2}\ln 2-\frac{1}{2}$

e. Divergent

Parts: $u=x \quad dv=e^{-x}\,dx$
$du=dx \quad v=-e^{-x}$

$$\int_0^{\ln 2} xe^{-x}\,dx=\left[-xe^{-x}+\int e^{-x}\,dx\right]_0^{\ln 2}=\left[-xe^{-x}-e^{-x}\right]_0^{\ln 2}=(-\ln 2e^{-\ln 2}-e^{-\ln 2})-(-1)=-\frac{1}{2}\ln 2+\frac{1}{2}$$

2. Compute $\int_0^{\pi/4}\cos\theta\sin^3\theta\,d\theta$

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{1}{8}$

d. $\frac{1}{16}$ CORRECT

e. $\frac{1}{32}$

$u=\sin\theta \qquad du=\cos\theta\,d\theta \qquad \int_0^{\pi/4}\cos\theta\sin^3\theta\,dx=\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/4}=\frac{1}{4}\frac{1}{\sqrt{2}^{\,4}}=\frac{1}{16}$

3. The partial fraction decomposition of $\frac{1}{x^2-x}$ is

a. $\frac{1}{x-1}+\frac{1}{x}$

b. $\frac{1}{x-1}-\frac{1}{x}$ CORRECT

c. $\frac{1}{x}-\frac{1}{x-1}$

d. $\frac{1}{x}+\frac{1}{x+1}$

e. $\frac{1}{x+1}-\frac{1}{x}$

$$\frac{1}{x^2-x}=\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1} \quad\Rightarrow\quad 1=A(x-1)+Bx=(A+B)x-A$$

$$\Rightarrow\quad A+B=0,\qquad -A=1\quad\Rightarrow\quad A=-1\qquad B=1\quad\Rightarrow\quad \frac{1}{x^2-x}=\frac{-1}{x}+\frac{1}{x-1}$$

4. Find the average value of $f(x) = \cos x$ on the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

a. $\frac{2\sqrt{2}}{\pi}$ CORRECT

b. $\frac{\sqrt{2}}{\pi}$

c. $\sqrt{2}$

d. $\frac{1}{\sqrt{2}}$

e. $\frac{\pi}{\sqrt{2}}$

$$f_{\text{ave}} = \frac{1}{\pi/2} \int_{-\pi/4}^{\pi/4} \cos x\, dx = \frac{2}{\pi}\Big[\sin x\Big]_{-\pi/4}^{\pi/4} = \frac{2}{\pi}\left(\frac{1}{\sqrt{2}}\right) - \frac{2}{\pi}\left(-\frac{1}{\sqrt{2}}\right) = \frac{4}{\pi\sqrt{2}} = \frac{2\sqrt{2}}{\pi}$$

5. The region bounded by $x = 0$, $x = \cos y$, $y = 0$, $y = \frac{\pi}{4}$ is rotated about the x-axis.
Which integral gives the volume of the solid of revolution?

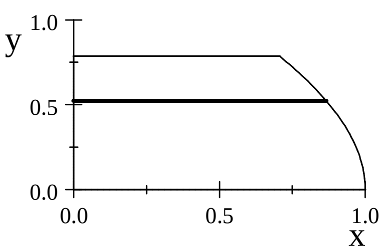
a. $\int_0^{\pi/4} 2\pi \cos^2 y\, dy$

b. $\int_0^{\sqrt{2}/2} 2\pi x \arccos x\, dx$

c. $\int_0^{\pi/4} 2\pi y \cos y\, dy$ CORRECT

d. $\int_0^{\sqrt{2}/2} \pi(\cos^2 x - x^2)\, dx$

e. $\int_0^{\pi/4} 2\pi y^2\, dy$



y-integral cylinders

$r = y$ $h = x = \cos y$

$$V = \int_0^{\pi/4} 2\pi r h\, dy = \int_0^{\pi/4} 2\pi y \cos y\, dy$$

6. The base of a solid is the semi-circle between $y = \sqrt{9 - x^2}$ and the x-axis.
The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

a. $\frac{9}{2}\pi$

b. 9π

c. 9

d. 18

e. 36 CORRECT

This is a x-integral. The side of the square is $s = \sqrt{9 - x^2}$. So the area is $A = s^2 = 9 - x^2$.

So the volume is $V = \int_{-3}^{3} A\, dx = \int_{-3}^{3} (9 - x^2)\, dx = \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2(27 - 9) = 36$

7. $\sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} =$

a. $\frac{9}{7}$

b. 4

c. 9 CORRECT

d. 3

e. Diverges

$a = \frac{3^2}{4^{2-1}} = \frac{9}{4} \qquad r = \frac{3}{4} \qquad |r| < 1 \qquad \sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} = \frac{\frac{9}{4}}{1 - \frac{3}{4}} = \frac{9}{4-3} = 9$

8. As n approaches infinity, the sequence $a_n = \frac{1 - \cos n}{n^2}$

a. converges to $-\frac{1}{2}$

b. converges to 0 CORRECT

c. converges to $\frac{1}{2}$

d. converges to 1

e. diverges

Since $0 \le \frac{1 - \cos n}{n^2} \le \frac{2}{n^2}$ and $\lim_{n\to\infty} \frac{2}{n^2} = 0$, the Squeeze Theorem says $\lim_{n\to\infty} \frac{1 - \cos n}{n^2} = 0$.

9. If $g(x) = \cos(x^2)$, what is $g^{(8)}(0)$, the 8th derivative at zero?
HINT: What is the coefficient of x^8 in the Maclaurin series for $\cos(x^2)$?

a. $\frac{1}{8 \cdot 7 \cdot 6 \cdot 5}$

b. $4!$

c. $\frac{1}{4!}$

d. $8 \cdot 7 \cdot 6 \cdot 5$ CORRECT

e. $\frac{1}{8!}$

On the one hand, the Maclaurin series for $\cos(t)$ is $\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$.

So the Maclaurin series for $\cos(x^2)$ is $\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$.

On the other hand the Maclaurin series for any function $g(x)$ is

$$g(x) = g(0) + g'(0)\,x + \cdots + \frac{g^{(8)}(0)}{8!}\,x^8 + \cdots$$

Since these must be equal, the coefficients of x^8 must be equal: $\frac{g^{(8)}(0)}{8!} = \frac{1}{4!}$

So $g^{(8)}(0) = \frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

10. Compute $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right)$

a. $-\frac{1}{2}$ CORRECT

b. $\frac{1}{2}$

c. 1

d. 2

e. Divergent

$$S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right) = \left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\cdots+\left(\frac{k}{k+1}-\frac{k+1}{k+2}\right) = \frac{1}{2}-\frac{k+1}{k+2}$$

$$S = \lim_{k\to\infty}\left(\frac{1}{2}-\frac{k+1}{k+2}\right) = -\frac{1}{2}$$

11. Compute $\lim_{x\to 0}\frac{\sin(2x)-2x}{3x^3}$

a. $-\frac{1}{9}$

b. -4

c. $-\frac{8}{9}$

d. $-\frac{4}{3}$

e. $-\frac{4}{9}$ CORRECT

$$\lim_{x\to 0}\frac{\sin(2x)-2x}{3x^3} = \lim_{x\to 0}\frac{\left[2x-\frac{(2x)^3}{3!}+\cdots\right]-2x}{3x^3} = \lim_{x\to 0}\left[-\frac{(2x)^3}{3!3x^3}+\cdots\right] = -\frac{8}{6\cdot 3} = -\frac{4}{9}$$

12. Estimate $\int_0^{0.1}\sin(x^2)\,dx$ to within an error of $|E| < 0.0001,$ HINT: Use a Maclaurin series.

a. $0.1-\frac{(0.1)^3}{6}$

b. 0.1

c. $(0.1)^2-\frac{(0.1)^6}{6}$

d. $\frac{(0.1)^3}{3}$ CORRECT

e. $(0.1)^2$

$\sin x = x-\frac{x^3}{6}+\cdots$ $\quad \sin x^2 = x^2-\frac{x^6}{6}+\cdots$ $\quad \int_0^x \sin(x^2)\,dx = \frac{x^3}{3}-\frac{x^7}{42}+\cdots$

Using 1 term: $\int_0^{0.1}\sin(x^2)\,dx \approx \frac{(0.1)^3}{3}$. The error is less than the next term

$|E| < \frac{(0.1)^7}{42} < 10^{-7}$

13. If $\vec{u}$ points North and $\vec{v}$ points South-East, then $\vec{u} \times \vec{v}$ points

- **a.** Up
- **b.** Down CORRECT
- **c.** East-North-East
- **d.** West-South-West
- **e.** North-West

Hold your right fingers North with your palm facing South-East. Your thumb points Down.

14. Find the area of a triangle with edges $\vec{a} = (3,-2,1)$ and $\vec{b} = (-1,0,1)$.

- **a.** 1
- **b.** 2
- **c.** $\sqrt{6}$ CORRECT
- **d.** 6
- **e.** $2\sqrt{6}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{\imath}(-2-0) - \vec{\jmath}(3+1) + \vec{k}(0-2) = (-2,-4,-2)$$

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{4+16+4} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

15. A vector $\vec{u}$ has length 5. A vector $\vec{v}$ has length 4. The angle between them is $60°$. Find $\vec{u} \cdot \vec{v}$.

- **a.** 10 CORRECT
- **b.** $\dfrac{1}{40}$
- **c.** $\dfrac{\sqrt{3}}{40}$
- **d.** 40
- **e.** $10\sqrt{3}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = 5 \cdot 4 \cdot \cos 60° = \frac{20}{2} = 10$$

Work Out (4 questions, 12 points each)

Show all you work.

16. Find the arc length of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ between $x = 1$ and $x = e$.

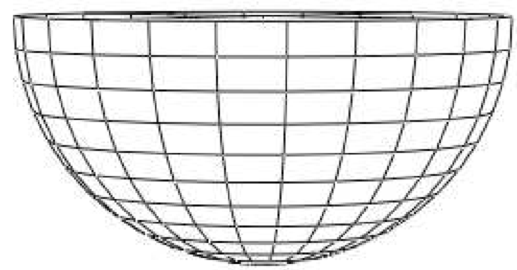
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right)dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \left(\frac{e^2}{4} + \frac{1}{2}\right) - \left(\frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

17. Find the work done to pump the water out the top of a hemispherical bowl of radius 5 cm if it is filled to the top.

The density of water is $\rho = 1\ \frac{\text{gm}}{\text{cm}^3}$.

The acceleration of gravity is $g = 980\ \frac{\text{cm}}{\text{sec}^2}$.



Measure y down from the top.

The slice at depth y must be lifted a distance $D = y$,

and has radius r which satisfies $r^2 + y^2 = 25$.

So this slice has volume $dV = \pi r^2\,dy = \pi(25 - y^2)\,dy$.

The force to lift this slice is its weight $dF = \rho g\,dV = \rho g \pi(25 - y^2)\,dy$.

So the work is

$$W = \int_0^5 D\,dF = \int_0^5 y\rho g\pi(25 - y^2)\,dy = \rho g\pi\int_0^5 (25y - y^3)\,dy = \rho g\pi\left[25\frac{y^2}{2} - \frac{y^4}{4}\right]_0^5$$

$$= \rho g\pi 5^4\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{\rho g\pi 5^4}{4} = \frac{980\pi 625}{4}$$

18. Compute $\displaystyle\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx.$

$x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$

$$\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx = \int_0^{\pi/4} \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta = 3\int_0^{\pi/4} \tan^2\theta\,d\theta$$

$$= 3\int_0^{\pi/4} \sec^2\theta - 1\,d\theta = 3\Big[\tan\theta - \theta\Big]_0^{\pi/4} = 3\left(1-\frac{\pi}{4}\right)$$

19. Find the radius and interval of convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^n}{4\ln n}$. Be sure to check the endpoints. Name or state any test you use and check the conditions.

Ratio Test: $\quad a_n = \dfrac{(x-3)^n}{4\ln n} \qquad a_{n+1} = \dfrac{(x-3)^{n+1}}{4\ln(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{4\ln(n+1)}\,\frac{4\ln n}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\ln n}{\ln(n+1)}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = |x-3| < 1$$

Radius: $R = 1.$

Converges if $2 < x < 4$.

At $x = 2$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{4\ln n}$ which converges by the Alternating Series Test.

The $(-1)^n$ says it alternates. The $\dfrac{1}{4\ln n}$ decreases and $\displaystyle\lim_{n\to\infty} \frac{1}{4\ln n} = 0$.

At $x = 4$ the series is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4\ln n}$. We apply the Comparison Test with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3n}$ which is a divergent harmonic series. Since $n > \ln n$ we have $\dfrac{1}{3n} < \dfrac{1}{3\ln n}$ and hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3\ln n}$ also diverges.

Interval: $[2,4)$.