MATH 152, SPRING 2012
COMMON EXAM II - VERSION A - SOLUTIONS

Last Name: ______________________________ **First Name**: ______________________________

Signature: ______________________________ **Section No**: _______________

PART I: **Multiple Choice** (**4 pts each**)

1. Find the sum of the geometric series $S = \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$.

a. $S = 3$
b. $S = 2$
c. $S = \frac{2}{3}$
d. $S = \frac{4}{15}$
e. $S = \frac{4}{3}$ Correct Choice

Solution: $a = \frac{4}{9}$ $r = \frac{2}{3}$ $S = \frac{\frac{4}{9}}{1 - 2/3} = \frac{4}{9-6} = \frac{4}{3}$

2. The improper integral $\int_1^e \frac{dx}{x \ln x}$

a. diverges to ∞. Correct Choice
b. diverges to $-\infty$.
c. converges to 1.
d. converges to -1.
e. converges to $\frac{1}{e} - 1$.

Solution: $u = \ln x$ $du = \frac{dx}{x}$ $\int_1^e \frac{dx}{x \ln x} = \int_0^1 \frac{du}{u} = \ln 1 - \ln 0 = 0 - -\infty = +\infty$

3. Which of the following integrals gives the surface area obtained by rotating the curve $y = e^{-4x}$, for $0 \le x \le 1$, about the y-axis?

a. $\int_0^1 2\pi e^{-4x} \sqrt{1 + 16e^{-8x}}\, dx$
b. $\int_0^1 2\pi x \sqrt{1 + 16e^{-8x}}\, dx$ Correct Choice
c. $\int_1^{e^{-4}} 2\pi y \sqrt{1 + \frac{1}{16y^2}}\, dy$
d. $\int_0^1 \frac{\pi}{2} \sqrt{16y^2 + 1}\, dy$
e. $\int_0^1 \frac{\pi}{8} \frac{\ln y}{y} \sqrt{16y^2 + 1}\, dy$

Solution: x-integral because $y = f(x)$ and $0 \le x \le 1$. $r = x$ because y-axis.

$$A = \int_0^1 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2\pi x \sqrt{1 + (-4e^{-4x})^2}\, dx = \int_0^1 2\pi x \sqrt{1 + 16e^{-8x}}\, dx$$

4. Compute $\int_{-1}^{\infty} \frac{dx}{1+x^2}$.

- **a**. $\frac{3\pi}{4}$ Correct Choice
- **b**. $\frac{\pi}{2}$
- **c**. $\frac{\pi}{4}$
- **d**. ∞
- **e**. 0

Solution: $\int_{-1}^{\infty} \frac{dx}{1+x^2} = \arctan(x)\Big|_{-1}^{\infty} = \frac{\pi}{2} - -\frac{\pi}{4} = \frac{3\pi}{4}$

5. By substituting $x = 3\tan\theta$, the integral $\int_0^3 x^2\sqrt{x^2+9}\,dx$ becomes

- **a**. $\int_0^{\pi/4} 27\tan^2\theta\sec\theta\,d\theta$
- **b**. $\int_0^{3} 27\tan^2\theta\sec^3\theta\,d\theta$
- **c**. $\int_0^{\pi/4} 81\tan^3\theta\sec^2\theta\,d\theta$
- **d**. $\int_0^{\pi/4} 81\tan^2\theta\sec^2\theta\,d\theta$
- **e**. $\int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$ Correct Choice

Solution: $x^2 = 9\tan^2\theta$ $\quad \sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta$ $\quad dx = 3\sec^2\theta\,d\theta$

$\int_0^{\pi/4} 9\tan^2\theta \cdot 3\sec\theta \cdot 3\sec^2\theta\,d\theta = \int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n + 2^n}{6^n} =$

- **a**. $\frac{1}{3}$
- **b**. $\frac{5}{14}$
- **c**. $\frac{33}{14}$ Correct Choice
- **d**. $\frac{3}{10}$
- **e**. $\frac{27}{10}$

Solution: $\sum_{n=0}^{\infty} \frac{(-1)^n + 2^n}{6^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^n} + \sum_{n=0}^{\infty} \frac{2^n}{6^n} = \frac{1}{1 - -\frac{1}{6}} + \frac{1}{1-\frac{1}{3}} = \frac{6}{7} + \frac{3}{2} = \frac{33}{14}$

7. Find the length of the curve $x = t^2, \quad y = t^3, \quad$ for $0 \le t \le 1$.

a. $\frac{1}{27}\left(13\sqrt{13} - 8\right)$ Correct Choice

b. $\frac{2\pi}{27}\left(13\sqrt{13} - 8\right)$

c. $\frac{1}{27}\left(13\sqrt{13} - 1\right)$

d. $\frac{1}{27}$

e. $\frac{2\pi}{27}\left(13\sqrt{13} - 1\right)$

Solution: $L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt$

$u = 4 + 9t^2 \qquad L = \frac{1}{18}\int_4^{13} \sqrt{u}\, du = \left[\frac{1}{18}\,\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}\left(13\sqrt{13} - 8\right)$

8. Which of the following series diverges by the Test for Divergence?

a. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

b. $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$ Correct Choice

c. $\sum_{n=1}^{\infty} \frac{n}{n!}$

d. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

e. The Test for Divergence is inconclusive for all of the above series.

Solution:

$\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{1}{n} = 0, \quad \lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0, \quad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$

$\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) = \sin\left(\frac{\pi}{2}\right) = 1$ So $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$ diverges.

9. The recursive sequence defined by $a_1 = 2, \quad a_{n+1} = 5 - \frac{4}{a_n}$ converges. Find the limit.

a. 1

b. 4 Correct Choice

c. 5

d. $\frac{5}{2}$

e. 2

Solution: $\lim_{n\to\infty} a_{n+1} = 5 - \frac{4}{\lim_{n\to\infty} a_n} \qquad L = 5 - \frac{4}{L} \qquad L^2 - 5L + 4 = 0 \qquad (L-4)(L-1) = 0$

Limit must be 1 or 4. $\quad a_1 = 2, \quad a_2 = 3, \quad a_1 = \frac{11}{3}, \quad a_n$ increasing from 2, Limit must be 4.

10. Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{dx}{e^x + \sqrt{x}}$?

a. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ converges.

b. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges.

c. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ diverges.

d. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ converges.

Correct Choice

e. The integral converges to 0.

Solution: $\frac{1}{e^x + \sqrt{x}} < \frac{1}{\sqrt{x}}$ and $\frac{1}{e^x + \sqrt{x}} < \frac{1}{e^x}$ So (b) and (c) are wrong.

$\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges and $\int_1^\infty \frac{dx}{e^x}$ converges So (a) and (c) are wrong.

$\frac{1}{e^x + \sqrt{x}} > 0$ So (e) is wrong. (d) is correct by the Comparison Test.

11. The sequence whose terms are $a_n = \frac{n^2 - 1}{n^2}$

a. decreases and converges to 1.

b. increases and converges to 1. Correct Choice

c. decreases and converges to 0.

d. increases and converges to 0.

e. diverges.

Solution: $\lim_{n\to\infty} \frac{n^2 - 1}{n^2} = \lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right) = 1$ $a_n = 1 - \frac{1}{n^2}$ is less than 1 but getting bigger.

12. Find the surface area obtained by rotating the curve $x = \cos(2t)$, $y = \sin(2t)$, for $0 \le t \le \frac{\pi}{4}$, about the x-axis.

a. $\frac{\pi}{4}$

b. 2π Correct Choice

c. $\frac{\pi}{2}$

d. π

e. 4π

Solution: $r = y = \sin(2t)$ because x-axis.

$$A = \int_0^{\pi/4} 2\pi r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/4} 2\pi \sin(2t) \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}\, dt$$

$$= \int_0^{\pi/4} 2\pi \sin(2t)\, 2\, dt = 4\pi\left[\frac{-\cos(2t)}{2}\right]_0^{\pi/4} = 2\pi\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) = 2\pi$$

13. $\displaystyle\int \frac{1}{x^2(x-1)}\,dx =$

a. $\ln|x-1| + \frac{1}{x} + C$
b. $\ln|x^2(x-1)| + C$
c. $\ln|x| - \frac{1}{x} - \ln|x-1| + C$
d. $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$ Correct Choice
e. $\ln|x-1| - \frac{1}{x} + C$

Solution: $\displaystyle \frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \qquad 1 = Ax(x-1) + B(x-1) + Cx^2$

$x = 0: \Rightarrow B = -1 \qquad x = 1: \Rightarrow C = 1 \qquad$ Coeff of x^2: $0 = A + C \Rightarrow A = -1$

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + C$$

PART II: **WORK OUT** (**48 points total**)

<u>**Directions**</u>: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

14. (10 pts) Integrate $\displaystyle\int \sqrt{16-9x^2}\,dx$.

Solution: $3x = 4\sin\theta \qquad 3\,dx = 4\cos\theta\,d\theta \qquad \sqrt{16-9x^2} = \sqrt{16-16\sin^2\theta} = 4\cos\theta$

$$\int \sqrt{16-9x^2}\,dx = \int 4\cos\theta\,\frac{4}{3}\cos\theta\,d\theta = \frac{16}{3}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{8}{3}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

Draw a triangle or: $\displaystyle \sin\theta = \frac{3x}{4} \qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9x^2}{16}}$

$\displaystyle \theta = \arcsin\frac{3x}{4} \qquad \frac{\sin 2\theta}{2} = \sin\theta\cos\theta = \frac{3x}{4}\sqrt{1-\frac{9x^2}{16}}$

$$\int \sqrt{16-9x^2}\,dx = \frac{8}{3}\left(\arcsin\frac{3x}{4} + \frac{3x}{4}\sqrt{1-\frac{9x^2}{16}}\right) + C$$

15. (8 pts) Find the sum of the series: $\displaystyle S = \sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right)$

Solution: $\displaystyle S_k = \sum_{n=1}^{k}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right)$

$$= \left(\cos\pi - \cos\frac{\pi}{2}\right) + \left(\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right) + \cdots + \left(\cos\frac{\pi}{k} - \cos\frac{\pi}{k+1}\right) = \cos\pi - \cos\frac{\pi}{k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\cos\pi - \cos\frac{\pi}{k+1}\right) = \cos\pi - \cos 0 = -1 - 1 = -2$$

16. (10 pts) Integrate $\displaystyle\int \frac{4x^2-1}{(x^2+1)(x-2)}\,dx$.

Solution: $\displaystyle\frac{4x^2-1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$ $\qquad 4x^2-1 = (Ax+B)(x-2)+C(x^2+1)$

$4x^2-1 = (Ax+B)(x-2)+C(x^2+1) = (A+C)x^2+(B-2A)x+(C-2B)$

$A+C=4 \qquad B-2A=0 \qquad C-2B=-1$

$C=4-A \qquad B=2A \qquad 4-A-4A=-1 \qquad 5A=5 \qquad A=1 \qquad B=2 \qquad C=3$

$$\int \frac{4x^2-1}{(x^2+1)(x-2)}\,dx = \int \frac{x+2}{x^2+1} + \frac{3}{x-2}\,dx = \int \frac{x}{x^2+1} + \frac{2}{x^2+1} + \frac{3}{x-2}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) + 2\arctan x + 3\ln|x-2| + C$$

17. If the n-th partial sum of the series $\displaystyle S = \sum_{n=1}^{\infty} a_n$ is given by $\displaystyle s_n = \frac{2n+1}{n}$,

(i) (5 pts) Find a_{10}.

Solution: $\displaystyle a_{10} = s_{10} - s_9 = \frac{2\cdot 10+1}{10} - \frac{2\cdot 9+1}{9} = \frac{21}{10} - \frac{19}{9} = -\frac{1}{90}$

(ii) (5 pts) Find the sum of the series $\displaystyle S = \sum_{n=1}^{\infty} a_n$.

Solution: $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2$

18. (10 pts) Find the surface area obtained by rotating the curve $y = \dfrac{x^2}{4} - \dfrac{1}{2}\ln x$, for $1 \le x \le 2$, about the y-axis.

Solution: x-integral because $y = f(x)$ and $1 \le x \le 2$. $\qquad r = x$ because y-axis.

$$A = \int_1^2 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1+\left(\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}\right)}$$

$$= \int_1^2 2\pi x\sqrt{\frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}}\,dx = \int_1^2 2\pi x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)\,dx = \pi\left[\frac{x^3}{3}+x\right]_1^2$$

$$= \pi\left(\frac{8}{3}+2\right) - \pi\left(\frac{1}{3}+1\right) = \frac{10}{3}\pi$$