MATH 152, SPRING 2012
COMMON EXAM II - VERSION B - SOLUTIONS

Last Name: ______________________________ **First Name**: ______________________________

Signature: ______________________________ **Section No**: _______________

PART I: **Multiple Choice** (**4 pts each**)

1. The sequence whose terms are $a_n = \dfrac{n^2 - 1}{n^2}$

a. increases and converges to 1. Correct Choice
b. decreases and converges to 1.
c. increases and converges to 0.
d. decreases and converges to 0.
e. diverges.

Solution: $\lim\limits_{n\to\infty} \dfrac{n^2-1}{n^2} = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{n^2}\right) = 1$ $a_n = 1 - \dfrac{1}{n^2}$ is less than 1 but getting bigger.

2. By substituting $x = 3\tan\theta$, the integral $\int_0^3 x^2\sqrt{x^2+9}\,dx$ becomes

a. $\int_0^{\pi/4} 81\tan^2\theta\sec^2\theta\,d\theta$
b. $\int_0^{\pi/4} 81\tan^3\theta\sec^2\theta\,d\theta$
c. $\int_0^{\pi/4} 27\tan^2\theta\sec\theta\,d\theta$
d. $\int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$ Correct Choice
e. $\int_0^{3} 27\tan^2\theta\sec^3\theta\,d\theta$

Solution: $x^2 = 9\tan^2\theta$ $\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$ $dx = 3\sec^2\theta\,d\theta$
$\int_0^{\pi/4} 9\tan^2\theta \cdot 3\sec\theta \cdot 3\sec^2\theta\,d\theta = \int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$

3. $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n + 2^n}{6^n} =$

a. $\dfrac{5}{14}$
b. $\dfrac{27}{10}$
c. $\dfrac{3}{10}$
d. $\dfrac{33}{14}$ Correct Choice
e. $\dfrac{1}{3}$

Solution: $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n + 2^n}{6^n} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{6^n} + \sum\limits_{n=0}^{\infty} \dfrac{2^n}{6^n} = \dfrac{1}{1 - -\frac{1}{6}} + \dfrac{1}{1 - \frac{1}{3}} = \dfrac{6}{7} + \dfrac{3}{2} = \dfrac{33}{14}$

4. Which of the following series diverges by the Test for Divergence?

a. $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$ Correct Choice

b. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

c. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

d. $\sum_{n=1}^{\infty} \frac{n}{n!}$

e. The Test for Divergence is inconclusive for all of the above series.

Solution:

$$\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{1}{n} = 0, \quad \lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0, \quad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$$

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \quad \text{So} \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) \text{ diverges.}$$

5. Find the length of the curve $x = t^2$, $y = t^3$, for $0 \le t \le 1$.

a. $\frac{2\pi}{27}\left(13\sqrt{13} - 1\right)$

b. $\frac{1}{27}$

c. $\frac{1}{27}\left(13\sqrt{13} - 1\right)$

d. $\frac{2\pi}{27}\left(13\sqrt{13} - 8\right)$

e. $\frac{1}{27}\left(13\sqrt{13} - 8\right)$ Correct Choice

Solution: $L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt$

$u = 4 + 9t^2 \qquad L = \frac{1}{18}\int_4^{13} \sqrt{u}\, du = \left[\frac{1}{18}\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}\left(13\sqrt{13} - 8\right)$

6. Find the surface area obtained by rotating the curve $x = \cos(2t)$, $y = \sin(2t)$, for $0 \le t \le \frac{\pi}{4}$, about the x-axis.

a. $\frac{\pi}{2}$

b. π

c. $\frac{\pi}{4}$

d. 4π

e. 2π Correct Choice

Solution: $r = y = \sin(2t)$ because x-axis.

$$A = \int_0^{\pi/4} 2\pi r\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/4} 2\pi\sin(2t)\sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}\, dt$$

$$= \int_0^{\pi/4} 2\pi\sin(2t)\, 2\, dt = 4\pi\left[\frac{-\cos(2t)}{2}\right]_0^{\pi/4} = 2\pi\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) = 2\pi$$

7. Find the sum of the geometric series $S = \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$.

a. $S = 2$
b. $S = 3$
c. $S = \frac{4}{3}$ Correct Choice
d. $S = \frac{4}{15}$
e. $S = \frac{2}{3}$

Solution: $a = \frac{4}{9}$ $r = \frac{2}{3}$ $S = \frac{\frac{4}{9}}{1 - 2/3} = \frac{4}{9-6} = \frac{4}{3}$

8. Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{dx}{e^x + \sqrt{x}}$?

a. The integral converges to 0.
b. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ converges.
Correct Choice
c. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges.
d. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ diverges.
e. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ converges.

Solution: $\frac{1}{e^x + \sqrt{x}} < \frac{1}{\sqrt{x}}$ and $\frac{1}{e^x + \sqrt{x}} < \frac{1}{e^x}$ So (c) and (d) are wrong.
$\int_1^\infty \frac{dx}{e^x}$ converges and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges So (d) and (e) are wrong.
$\frac{1}{e^x + \sqrt{x}} > 0$ So (a) is wrong. (b) is correct by the Comparison Test.

9. The recursive sequence defined by $a_1 = 2$, $a_{n+1} = 5 - \frac{4}{a_n}$ converges. Find the limit.

a. 4 Correct Choice
b. 5
c. 1
d. $\frac{5}{2}$
e. 2

Solution: $\lim_{n\to\infty} a_{n+1} = 5 - \frac{4}{\lim_{n\to\infty} a_n}$ $L = 5 - \frac{4}{L}$ $L^2 - 5L + 4 = 0$ $(L-4)(L-1) = 0$
Limit must be 1 or 4. $a_1 = 2$, $a_2 = 3$, $a_1 = \frac{11}{3}$, a_n increasing from 2, Limit must be 4.

10. $\int \frac{1}{x^2(x-1)}\,dx =$

a. $\ln|x-1| - \frac{1}{x} + C$
b. $\ln|x| - \frac{1}{x} - \ln|x-1| + C$
c. $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$ Correct Choice
d. $\ln|x-1| + \frac{1}{x} + C$
e. $\ln|x^2(x-1)| + C$

Solution: $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ $\quad 1 = Ax(x-1) + B(x-1) + Cx^2$
$x = 0: \Rightarrow B = -1 \quad x = 1: \Rightarrow C = 1 \quad$ Coeff of x^2: $0 = A + C \Rightarrow A = -1$
$\int \frac{1}{x^2(x-1)}\,dx = \int\left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + C$

11. Compute $\int_{-1}^{\infty} \frac{dx}{1+x^2}$.

a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$
c. $\frac{3\pi}{4}$ Correct Choice
d. ∞
e. 0

Solution: $\int_{-1}^{\infty} \frac{dx}{1+x^2} = \arctan(x)\Big|_{-1}^{\infty} = \frac{\pi}{2} - -\frac{\pi}{4} = \frac{3\pi}{4}$

12. Which of the following integrals gives the surface area obtained by rotating the curve $y = e^{-4x}$,
for $0 \le x \le 1$, about the y-axis?

a. $\int_0^1 2\pi x\sqrt{1+16e^{-8x}}\,dx$ Correct Choice
b. $\int_0^1 2\pi e^{-4x}\sqrt{1+16e^{-8x}}\,dx$
c. $\int_1^{e^{-4}} 2\pi y\sqrt{1+\frac{1}{16y^2}}\,dy$
d. $\int_0^1 \frac{\pi}{2}\sqrt{16y^2+1}\,dy$
e. $\int_0^1 \frac{\pi}{8}\frac{\ln y}{y}\sqrt{16y^2+1}\,dy$

Solution: x-integral because $y = f(x)$ and $0 \le x \le 1$. $\quad r = x$ because y-axis.
$A = \int_0^1 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+(-4e^{-4x})^2}\,dx = \int_0^1 2\pi x\sqrt{1+16e^{-8x}}\,dx$

13. The improper integral $\int_1^e \frac{dx}{x\ln x}$

a. diverges to $-\infty$.
b. diverges to ∞. Correct Choice
c. converges to -1.
d. converges to 1.
e. converges to $\frac{1}{e} - 1$.

Solution: $u = \ln x \qquad du = \frac{dx}{x} \qquad \int_1^e \frac{dx}{x\ln x} = \int_0^1 \frac{du}{u} = \ln 1 - \ln 0 = 0 - -\infty = +\infty$

PART II: **WORK OUT** (**48 points total**)

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

14. (10 pts) Integrate $\int \sqrt{9 - 16x^2}\, dx$.

Solution: $4x = 3\sin\theta \qquad 4\,dx = 3\cos\theta\, d\theta \qquad \sqrt{9-16x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$

$$\int \sqrt{9-16x^2}\, dx = \int 3\cos\theta\, \frac{3}{4}\cos\theta\, d\theta = \frac{9}{4}\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{9}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

Draw a triangle or: $\sin\theta = \frac{4x}{3} \qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1 - \frac{16x^2}{9}}$

$$\theta = \arcsin\frac{4x}{3} \qquad \frac{\sin 2\theta}{2} = \sin\theta\cos\theta = \frac{4x}{3}\sqrt{1-\frac{16x^2}{9}}$$

$$\int \sqrt{9-16x^2}\, dx = \frac{9}{8}\left(\arcsin\frac{4x}{3} + \frac{4x}{3}\sqrt{1-\frac{16x^2}{9}}\right) + C$$

15. (8 pts) Find the sum of the series: $S = \sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right)$

Solution: $S_k = \sum_{n=1}^{k}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right)$

$$= \left(\cos\pi - \cos\frac{\pi}{2}\right) + \left(\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right) + \cdots + \left(\cos\frac{\pi}{k} - \cos\frac{\pi}{k+1}\right) = \cos\pi - \cos\frac{\pi}{k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\cos\pi - \cos\frac{\pi}{k+1}\right) = \cos\pi - \cos 0 = -1 - 1 = -2$$

16. (10 pts) Integrate $\displaystyle\int \frac{4x^2-1}{(x^2+1)(x-2)}\,dx$.

Solution: $\displaystyle\frac{4x^2-1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$ $\qquad 4x^2-1 = (Ax+B)(x-2)+C(x^2+1)$

$4x^2-1 = (Ax+B)(x-2)+C(x^2+1) = (A+C)x^2+(B-2A)x+(C-2B)$

$A+C = 4 \qquad B-2A = 0 \qquad C-2B = -1$

$C = 4-A \qquad B = 2A \qquad 4-A-4A = -1 \qquad 5A = 5 \qquad A = 1 \qquad B = 2 \qquad C = 3$

$$\int \frac{4x^2-1}{(x^2+1)(x-2)}\,dx = \int \frac{x+2}{x^2+1} + \frac{3}{x-2}\,dx = \int \frac{x}{x^2+1} + \frac{2}{x^2+1} + \frac{3}{x-2}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) + 2\arctan x + 3\ln|x-2| + C$$

17. If the n-th partial sum of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is given by $s_n = \dfrac{n+2}{2n}$,

(i) (5 pts) Find a_{10}.

Solution: $\displaystyle a_{10} = s_{10} - s_9 = \frac{10+2}{2\cdot 10} - \frac{9+2}{2\cdot 9} = \frac{6}{10} - \frac{11}{18} = -\frac{1}{90}$

(ii) (5 pts) Find the sum of the series $\displaystyle S = \sum_{n=1}^{\infty} a_n$.

Solution: $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2}$

18. (10 pts) Find the surface area obtained by rotating the curve $y = \dfrac{x^2}{4} - \dfrac{1}{2}\ln x$, for $1 \le x \le 2$, about the y-axis.

Solution: x-integral because $y = f(x)$ and $1 \le x \le 2$. $\qquad r = x$ because y-axis.

$$A = \int_1^2 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1+\left(\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}\right)}$$

$$= \int_1^2 2\pi x\sqrt{\frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}}\,dx = \int_1^2 2\pi x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)\,dx = \pi\left[\frac{x^3}{3}+x\right]_1^2$$

$$= \pi\left(\frac{8}{3}+2\right) - \pi\left(\frac{1}{3}+1\right) = \frac{10}{3}\pi$$