Name______________________________ Section____________

MATH 152 FINAL EXAM Version A Spring 2016

Sections 555-557 Solutions P. Yasskin

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16	/ 5
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Multiple Choice: (13 problems, 4 points each)

1. **Average Value of a Function**

New Problem or Modify or Make Your Own Problem

Find the average value of the function f(x) = sin(x) on the interval [a,b] = [0,Pi].

a. $\frac{2}{\pi}$ correct choice

b. $\frac{1}{\pi}$

c. 2π

d. 2

e. 1

Solution: $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx = -\frac{1}{\pi}\cos(x)\Big|_0^\pi = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

2. **Integrals Which are Improper at an Endpoint**

New Problem

Problem Statement:

Determine if the following improper integral is convergent or divergent.

$$\int_{-2}^{\infty} (x+4)^{\frac{-1}{3}}\,dx$$

If convergent, compute it.

If divergent, determine if it is + infinity, - infinity, or neither.

a. converges to $\frac{3}{2^{1/3}}$

b. converges to $-\frac{3}{2^{1/3}}$

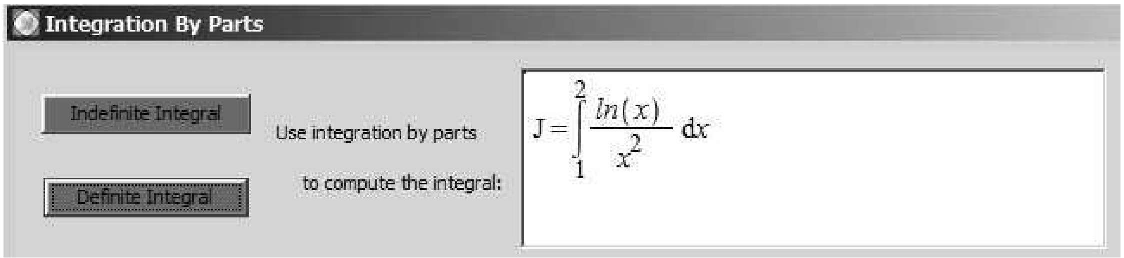
c. diverges to ∞ correct choice

d. diverges to $-\infty$

e. diverges but not to $\pm\infty$

Solution: $\int_{-2}^{\infty}(x+4)^{-1/3}\,dx = \frac{3(x+4)^{2/3}}{2}\Big|_{-2}^{\infty} = \infty - \frac{3(2)^{2/3}}{2} = \infty$

3.



a. $\frac{3 - \ln(2)}{2}$

b. $\frac{\ln(2) - 3}{2}$

c. $\frac{\ln(2) - 1}{2}$

d. $\frac{-\ln(2)}{2}$

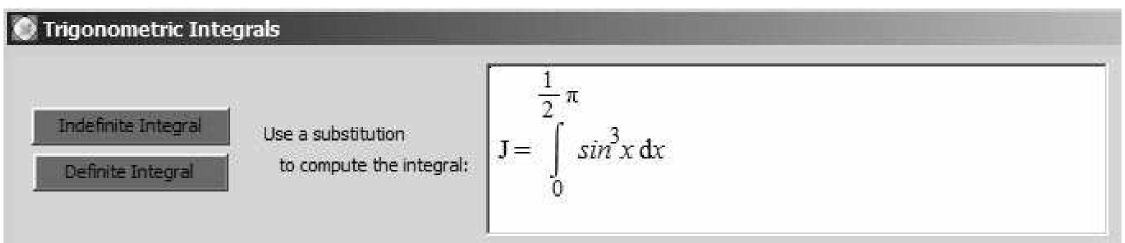
e. $\frac{1 - \ln(2)}{2}$ correct choice

Solution:

$u = \ln(x)$	$dv = \frac{1}{x^2}dx$
$du = \frac{1}{x}dx$	$v = \frac{-1}{x}$

$$J = \left[\frac{-\ln(x)}{x} - \int \frac{-1}{x^2}dx\right]_1^2 = \left[\frac{-\ln(x)}{x} - \frac{1}{x}\right]_1^2$$
$$= \left(\frac{-\ln(2)}{2} - \frac{1}{2}\right) - \left(\frac{-\ln(1)}{1} - \frac{1}{1}\right) = \frac{-\ln(2)}{2} + \frac{1}{2}$$

4.



a. $-\frac{1}{4}$

b. $\frac{1}{4}$

c. $\frac{2}{3}$ correct choice

d. $-\frac{4}{3}$

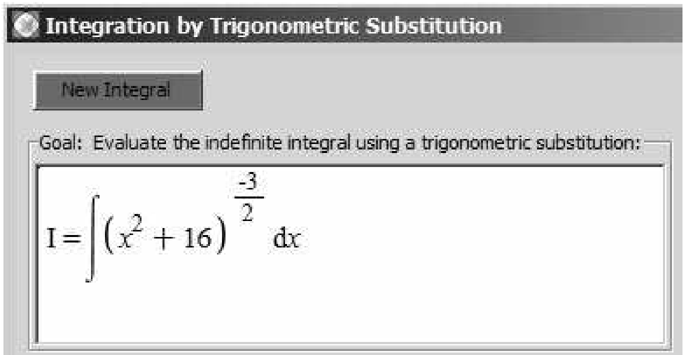
e. $\frac{4}{3}$

Solution:

$u = \cos x$
$du = -\sin x\,dx$

$$J = \int_0^{\pi/2}(1 - \cos^2 x)\sin x\,dx = -\int_1^0 (1 - u^2)\,du = \left[-u + \frac{u^3}{3}\right]_1^0$$
$$= -\left(-1 + \frac{1}{3}\right) = \frac{2}{3}$$

5.



a. $\frac{1}{16} \int \csc^2\theta \, d\theta$

b. $\frac{1}{64} \int \sec^2\theta \, d\theta$

c. $\frac{1}{16} \int \sin^3\theta \, d\theta$

d. $\frac{1}{16} \int \cos\theta \, d\theta$ correct choice

e. $\frac{1}{64} \int \cos^3\theta \, d\theta$

Simply identify the integral after the substitution.

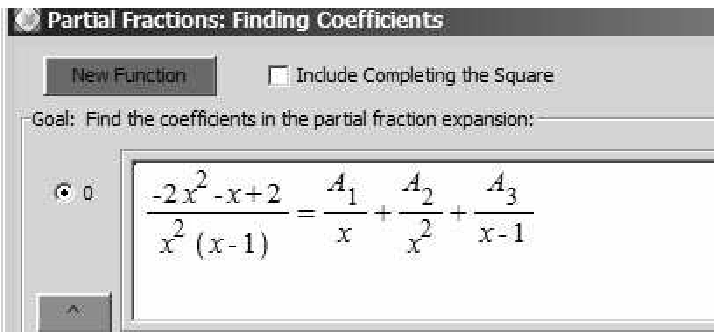
Solution:

$x = 4\tan\theta$
$dx = 4\sec^2\theta \, d\theta$

$$I = \int \frac{4\sec^2\theta \, d\theta}{(16\tan^2\theta + 16)^{3/2}} = \int \frac{4\sec^2\theta \, d\theta}{64(\sec^2\theta)^{3/2}} = \int \frac{d\theta}{16\sec\theta}$$

$$= \frac{1}{16} \int \cos\theta \, d\theta$$

6.



a. $A_1 = -1 \quad A_2 = -2$ correct choice

b. $A_1 = 1 \quad A_2 = 2$

c. $A_1 = -2 \quad A_2 = -1$

d. $A_1 = 2 \quad A_2 = 1$

e. $A_1 = -2 \quad A_2 = 1$

Just find A_1 and A_2.

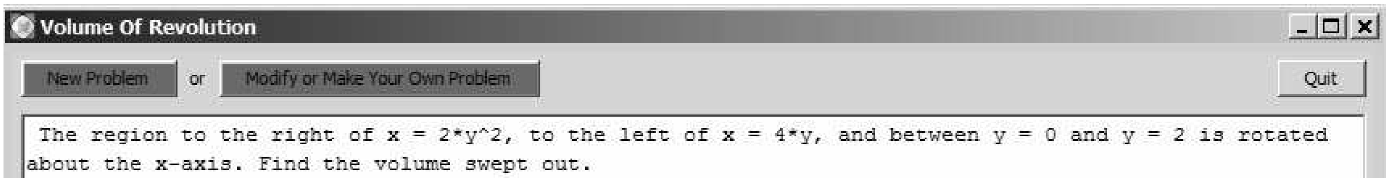
Solution: Clear the denominator: $-2x^2 - x + 2 = A_1 x(x-1) + A_2(x-1) + A_3 x^2$ (*)

Plug in $x = 0$: $2 = A_2(-1) \qquad A_2 = -2$

Differentiate (*): $-4x - 1 = A_1(2x-1) + A_2 + A_3 2x$

Plug in $x = 0$: $-1 = A_1(-1) + A_2 = -A_1 - 2 \qquad A_1 = -1$

7.



a. $\frac{8}{3}\pi$

b. $\frac{16}{3}\pi$ correct choice

c. $\frac{256}{3}\pi$

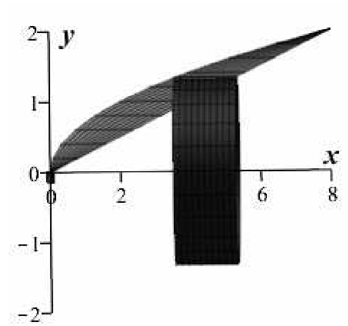
d. $\frac{16}{15}\pi$

e. $\frac{256}{15}\pi$

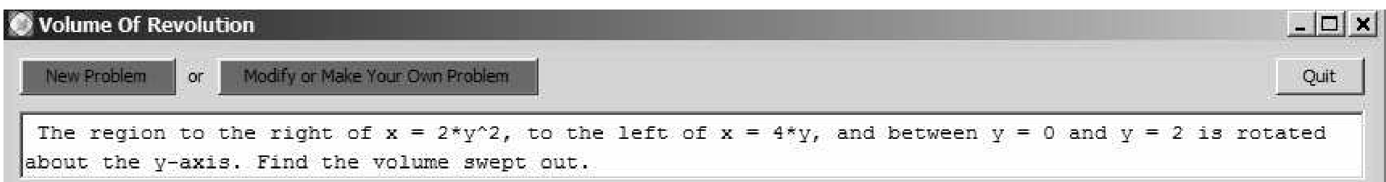
Solution: The region is shown. It is a y-integral. The horizontal slices rotate into cylinders.

$$V = \int_0^2 2\pi r h\, dy = \int_0^2 2\pi(y)(4y - 2y^2)\, dy$$

$$= 2\pi\left[\frac{4y^3}{3} - \frac{2y^4}{4}\right]_0^2 = 2\pi\left(\frac{32}{3} - 8\right) = \frac{16}{3}\pi$$



8.



a. $\frac{8}{3}\pi$

b. $\frac{16}{3}\pi$

c. $\frac{256}{3}\pi$

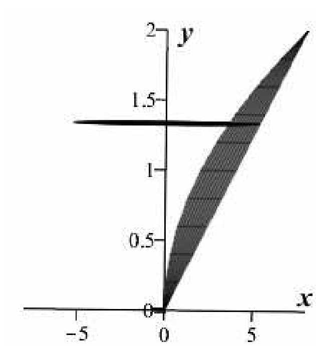
d. $\frac{16}{15}\pi$

e. $\frac{256}{15}\pi$ correct choice

Solution: The region is shown. It is a y-integral. The horizontal slices rotate into washers.

$$V = \int_0^2 \pi(R^2 - r^2)\, dy = \int_0^2 \pi(16y^2 - 4y^4)\, dy$$

$$= \pi\left[\frac{16y^3}{3} - \frac{4y^5}{5}\right]_0^2 = \pi\frac{5 \cdot 128 - 3 \cdot 128}{15} = \frac{256}{15}\pi$$



9. **Surface Area Of Solid Of Revolution**

New Problem or Modify or Make Your Own Problem

The curve y = 2/3*x^2, between x = 0 and x = 1, is rotated about the y-axis. Find the surface area of the surface of revolution.

a. $\frac{49}{36}$

b. $\frac{49}{72}$

c. $\frac{49}{144}$

d. $\frac{49}{36}\pi$ correct choice

e. $\frac{126}{72}\pi$

Solution: $L = \int_0^1 2\pi r\,ds = \int_0^1 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+\left(\frac{4}{3}x\right)^2}\,dx$

$= \int_0^1 \frac{2\pi}{3}x\sqrt{9+16x^2}\,dx = \frac{2\pi}{3}\left[\frac{2(9+16x^2)^{3/2}}{3\cdot 32}\right]_0^1 = \frac{\pi}{72}\left[(9+16x^2)^{3/2}\right]_0^1$

$= \frac{\pi}{72}25^{3/2} - \frac{\pi}{72}9^{3/2} = \frac{\pi}{72}(125-27) = \frac{49}{36}\pi$

10. **Work to Lift an Object with a Rope**

New Problem

Goal:

Find the work needed to lift a 12 lb object up a 50 ft building using a rope whose density is 4 lb/ft.

a. 5600 ft-lb correct choice

b. 5000 ft-lb

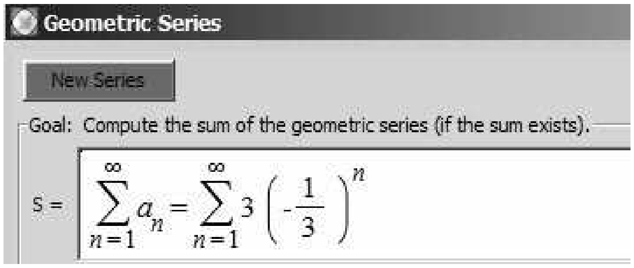
c. 3100 ft-lb

d. 2500 ft-lb

e. 600 ft-lb

Solution: The work to lift just the 12 lb weight is $W_1 = FD = 12\,\text{lb}\cdot 50\,\text{ft} = 600$ ft-lb.
Measuring y from the bottom of the building, the work to lift just the rope is
$W_2 = \int_0^{50} D\,dF = \int_0^{50}(50-y)\,4\,dy = [200y - 2y^2]_0^{50} = 5000$ ft-lb.
So the total work is $W = 600 + 5000 = 5600$ ft-lb

11.



a. $-\frac{3}{2}$

b. $-\frac{3}{4}$ correct choice

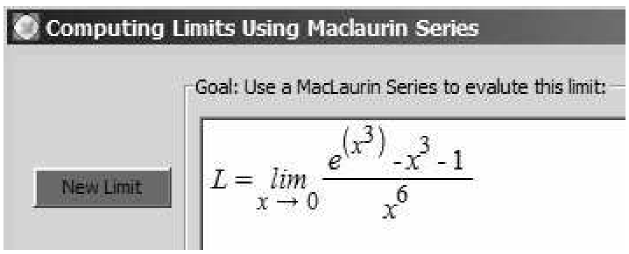
c. $\frac{9}{4}$

d. $\frac{9}{2}$

e. diverges

Solution: $a = 3\left(-\frac{1}{3}\right) = -1 \qquad r = -\frac{1}{3} \qquad S = \frac{a}{1-r} = \frac{-1}{1+\frac{1}{3}} = -\frac{3}{4}$

12.



a. $\frac{1}{2}$ correct choice

b. $\frac{1}{3}$

c. $\frac{1}{6}$

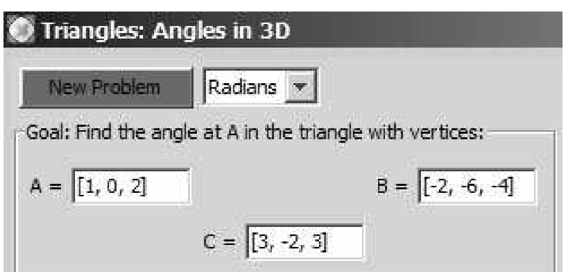
d. $\frac{1}{6!}$

e. diverges

Solution: $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots \qquad e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots$

$$L = \lim_{x \to 0} \frac{1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots - x^3 - 1}{x^6} = \lim_{x \to 0}\left(\frac{1}{2} + \frac{x^3}{6} + \cdots\right) = \frac{1}{2}$$

13.



a. 0

b. $\frac{\pi}{2}$ correct choice

c. $\arccos\left(\frac{2}{3}\right)$

d. $\arccos\left(\frac{2}{9}\right)$

e. $\arccos\left(\frac{4}{9}\right)$

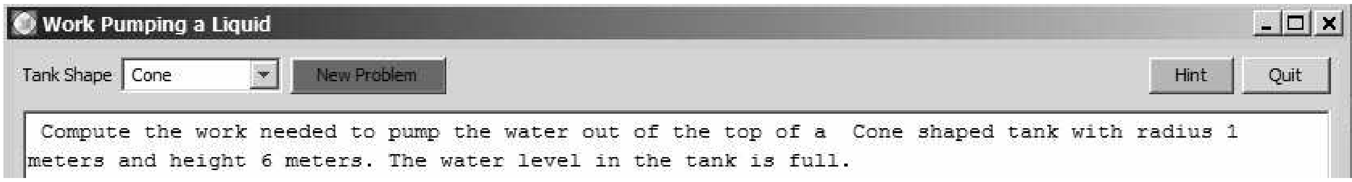
Solution: $\overrightarrow{AB} = (-3, -6, -6) \qquad \overrightarrow{AC} = (2, -2, 1)$

$$\left|\overrightarrow{AB}\right| = \sqrt{9 + 36 + 36} = 9 \qquad \left|\overrightarrow{AC}\right| = \sqrt{4 + 4 + 1} = 3 \qquad \overrightarrow{AB} \bullet \overrightarrow{AC} = -6 + 12 - 6 = 0$$

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{0}{9 \bullet 3} = 0 \qquad \theta = \frac{\pi}{2}$$

Work Out (5 questions, Points indicated. Show all you work.)

14. (20 points)



Write your answer as a multiple of ρg where ρ is the density of water and g is the acceleration of gravity. The vertex of the cone is at the bottom.

Solution: Put $y = 0$ at the bottom of the tank. The slice at height y is lifted a distance $D = 6 - y$.

It is a thin disk of radius r satisfying $\frac{r}{y} = \frac{1}{6}$. So $r = \frac{1}{6}y$. The area of the disk is $A = \pi r^2 = \frac{\pi y^2}{36}$.

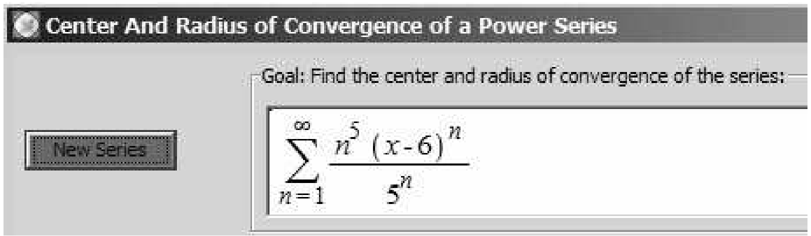
The volume of the disk is $dV = \frac{\pi y^2}{36}\,dy$. The weight of the disk is $dF = \rho g\,dV = \rho g\frac{\pi y^2}{36}\,dy$.

So the work is

$$W = \int D \cdot dF = \frac{\pi\rho g}{36}\int_0^6 (6-y)y^2\,dy = \frac{\pi\rho g}{36}\left[2y^3 - \frac{y^4}{4}\right]_0^6 = \frac{\pi\rho g}{36}\left(2\cdot 6^3 - \frac{6^4}{4}\right)$$

$$= 6\pi\rho g\left(2 - \frac{6}{4}\right) = 3\pi\rho g$$

15. (20 points)



Also find the interval of convergence by checking the endpoints.

a. (2 pts) Identify the center: $a =$ __6__

b. (8 pts) Find the radius of convergence:

Solution: Apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^5 |x-6|^{n+1}}{5^{n+1}} \frac{5^n}{n^5 |x-6|^n} = \frac{|x-6|}{5} \lim_{n\to\infty} \frac{(n+1)^5}{n^5} = \frac{|x-6|}{5} < 1$$

$|x-6| < 5 \qquad 1 < x < 11$ $R =$ __5__

c. (8 pts) Check the endpoints:

Solution:

$x = 1$: $\sum_{n=1}^{\infty} \frac{n^5(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n^5 \qquad \lim_{n\to\infty} (-1)^n n^5 =$ divergent $\neq 0$

Diverges by the n^{th}-Term Divergence Test

$x = 11$: $\sum_{n=1}^{\infty} \frac{n^5(5)^n}{5^n} = \sum_{n=1}^{\infty} n^5 \qquad \lim_{n\to\infty} n^5 = \infty \neq 0$

Diverges by the n^{th}-Term Divergence Test

d. (2 pts) Summarize the interval of convergence: $I =$ __$(1, 11)$__

16. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is absolutely convergent, convergent but not absolutely or divergent. Explain all tests you use.

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent by the Alternating Series Test since the $(-1)^n$ says it is alternating, $\frac{1}{n^{1/3}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent but not absolutely.

17. (5 points) The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

If it is approximated by its 100^{th} partial sum S_{100}, compute the integral bound on the error in this approximation.

Solution: The bound is

$$|E_7| = |S - S_{100}| < \int_{100}^{\infty} \frac{1}{n^2+1}\,dn = \Big[\arctan(n)\Big]_{100}^{\infty} = \frac{\pi}{2} - \arctan(100) \qquad (\approx 0.01)$$

18. (5 points) Compute the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}}$.

Solution: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$